https://doi.org/10.22364/atee.2022.44

Fostering Teachers' Mathematical Competence in Problem Solving

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ABSTRACT

The problem of a growing shortage of qualified mathematics teachers, which is emerging in many countries today, is also quite acute in Latvia. Moreover, there are only few students in Latvia who want to study mathematics at a serious level and become mathematics teachers. A shortage of students and teachers is one of the reasons for the decline in the quality of mathematics teaching. Teaching and learning mathematics is impossible without understanding and problem solving, which, as noted by the famous mathematician Paul Halmos, is 'the heart of mathematics'. It is a well-known idea that by developing problem-solving skills, we learn not only how to tackle mathematical problems, but also how to logically work our way through any problems we may face. Unfortunately, this idea does little to contribute to a successful tackling of the teacher shortage, which is not a mathematics problem for which somebody can find a quick and easy solution. This article offers some research topics that may be useful for teachers working with their gifted pupils; deals with a non-trivial problem of constructing magic polygons on a triangular lattice, which is recommended as a research topic for teachers to encourage them and their students to better master different solution strategies and so-called big ideas in mathematics; the article also gives some insights into mathematics teacher training education in Latvia.

Keywords: polyforms, polyiamond, polyomino, problem solving, research topics, teacher shortage

Introduction

Nowadays we can hear more and more about growing teacher shortage problems all over the world (Lyons, 2021; McLean Davies & Watterston, 2022; Natanson, 2022).

The shortage of teachers in Latvia and the problems in education are indicated by headlines in newspapers, media and, of course, on the Internet. Here are English translations of some of the most striking headlines (2022):

- Teacher shortage in STEM subjects threatens Latvia's future,
- Where teachers disappear or the "black hole" of the Latvian education system,
- Schools face a shortage of teachers and the biggest shortage – mathematics teachers (A large number of teacher vacancies – 1,000 – have been announced.),
- Many schools face a catastrophic shortage of teachers ahead of the new school year (The number of vacancies has already reached around 2,000, so for many schools it will be impossible to find the teachers they need for September 1st and possibly longer. Many schools will therefore have to cope with missing lessons, merged classes and overworked teachers in the new school year.).

At the end of August 2022 on the website *esiskolotajs.lu.lv* we can find information about 35 free mathematics teacher vacancies and on *izglitiba.riga.lv* we can see that in Riga, the capital of Latvia, there are 20 mathematics teachers' vacancies in need to be filled. In addition, all this is right before the new school year starts. According to the Teacher Vacancies website, there are currently 40 mathematics teacher vacancies. The shortage of teachers in schools is a problem that has persisted for years. There are several reasons for this, based on a variety of sources:

- the low prestige of the profession,
- the unbalanced salaries and workload,
- the hasty introduction of new curricula without adequate provision of teaching resources,
- parents' attitudes, teacher burnout, emotional abuse by students and parents,
- teachers have responsibilities, but no rights to take firm action against those who bully the whole class,
- the inability of the Ministry of Education and Science (Latvia) to find timely solutions to several issues and ineffective communication is the reason why the problems have become so acute.

In Latvia, mathematics teacher profession can be obtained in the University of Latvia, Liepaja University and Daugavpils University. It should be noted that in 2022, the University of Latvia received only 17 applications for the Mathematics Teacher study programme. The other two universities offer the Mathematics Teacher Study Programme every second year, not every year, due to the low number of applications. To make matters worse, there are students who decide to drop out of university, a significant number of graduates do not go to work in schools at all, and many future teachers are not sufficiently prepared to work with gifted pupils, to train them for mathematics olympiads or to supervise their research work. The quality of education is also affected by the fact that there are too many higher education institutions in Latvia, which significantly reduces competitiveness. Higher education institutions are trying to fill all budget places, first, but they do not have enough capacity to raise the quality of higher education. In 2019, Ilga Šuplinska, Minister of Education (from 2019 to 2021), emphasised that the number of higher education institutions in Latvia is not optimal, as evidenced, for example, by the fact that their infrastructure allows them to enrol around 160 000 students, but only 80 000 study.

The problem of teacher training and development has been an issue for many years. In Notices of the AMS (Wu, 1999) particularly emphasises that the only way to achieve better mathematics education is to have better mathematics teachers, and that "Any improvement in education must start with improvement of the teachers already in the classroom". To ensure that gifted pupils (future potential students at universities) are challenged enough we need to foster their teacher's mathematical competence in problem solving, so they can train and inspire their pupils to learn mathematics in greater depth. Although there are opportunities for teachers to develop their competence in school related topics by applying for courses, the courses that are oriented on developing more advanced skills (how to run mathematics clubs, how to prepare pupils for mathematics competitions, how to choose and develop appropriate research topics, etc.) are rare. That is why this paper mainly focuses on giving teachers some topics they can use to work at a higher level with the gifted pupils with a goal to supervise their scientific research.

Problem solving

Many of the commonly used terms, especially if they are non-mathematical, mean different things to different people. Albrecht, (2022) in her internet article notes "Problem solving is one example. The lack of a shared understanding complicates discussions at all levels, including national curriculum stoushes like the recent ones in Australia." This Internet article summarises different aspects of problem solving and provides important references such as, (Halmos, 1980; Polya, 2004), seminal paper of Alan Schoenfeld (Schoenfeld, 1992). To avoid ambiguity, let us clarify right away that we shall be dealing with mathematical problems, not routine school exercises, but non-trivial so-called challenging problems that can be useful as research topics for working with gifted pupils.

It is useful to know, that problem solving is the heart of mathematics. All the mathematics (theories, theorems, constructions etc.) we have now has been invented to solve some problems. Some of these problems might have come from science, economics, or even real-life situations while others are purely mathematical. In (Halmos et al., 1975) it is written

> *The best way to learn is to do, (...) What mathematics is really all about is solving concrete problems. (...) A good teacher challenges, asks, annoys, irritates, and maintains high standards – all that is generally not pleasant. A good teacher may not be a popular teacher (except perhaps with his ex-students), because some students don't like to be challenged, asked, annoyed, and irritated – but he produces pearls (instead of casting them in the proverbial manner).*

Mathematics popularisers, those interested in recreational mathematics know that one of the best ways to capture the imagination of young people and get them interested in mathematics is by "hooking them" on irresistible problems.

In mathematics, there are a variety of topics (unexplored islands) that gifted pupils could make a new contribution to, which are not only unfamiliar only to them but also to their teachers. A gifted pupil may occasionally feel frustrated that his teachers themselves are not prepared to deal with such olympiad-level problems. You cannot teach what you do not understand. As Marilyn Burns write (Burns, 2007, p. 5) "teachers can't teach for understanding if they don't have a firm foundation of understanding themselves".

In Book Series: Problem Solving in Mathematics and Beyond, Volume 22, Seduced by Mathematics Pritsker (2021) presents a problem solving process consisting of 8 steps and gives some problems as an example for reader to better understand how these steps work and are carried out. He refers to the 1973 edition of Polya's book "How to Solve It", and it should be noted that this book, a bestseller, has many editions in different languages, the first edition having been published in 1945. Polya (2004) in page 1 of his book "How to Solve It" has written that "One of the most important tasks of the teacher is to help his students. This task is not quite easy; it demands time, practice, devotion, and sound principles." Generations of readers have relished Polya's – indeed, brilliant – instructions on stripping away irrelevancies and going straight to the heart of the problem. Polya identifies four basic principles for problem solving:

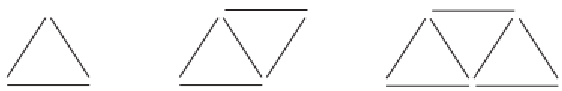
1. Understand the problem,
2. Devise a plan,
3. Carry out the plan,
4. Look back.

As it can be read in Polya's book, there are many reasonable ways to solve problems, but the skill of choosing an appropriate strategy is best learned best of all by solving mathematical problems. He mentions problem solving strategies such as guess and check, make an orderly list, eliminate possibilities, use symmetry, consider special cases, use direct reasoning, solve an equation, look for a pattern, draw a picture, solve a simpler problem, use model, work backwards, use formula, be ingenious. For challenging problems, see, for example, Problem 1 below, almost all of these strategies can be useful, but for typical school problems a small number of simple techniques may suffice.

Let us start with an instructive school exam task (VISC, 2022) that Form 9 pupils had to solve this year in Latvia, see Figure 1, where formulation was not mathematically correct and could have caused some confusion to pupils. This example illustrates that many teachers, even the so-called experts who design the tests, should practise, and develop not only their problem solving skills, but also even their mathematical problem formulation skills.

Eksāmens matemātikā 9. klasei Skolēna darba lapa 2. daļa 2022

8. uzdevums *(7 punkti).*
Zaiga no vienādiem 2,5 cm gariem kociņiem veido figūras (skat. zīm.).



1. figūra 2. figūra 3. figūra

8.1. Cik kociņu būs nepieciešams, lai izveidotu ceturto figūru?

Figure 1. Problem 8 from the examination in mathematics for Form 9 in Latvia, 2022

The translation of this school task is as follows:

Zaiga is making shapes (see. Figure 1) from sticks that are 2.5 cm long.

8.1. How many sticks will be needed to make the fourth shape?

8.2. Write an expression and calculate how many sticks will be needed to make the 100th figure.

8.3. Calculate the number of the figure the perimeter of which is 105 cm.

8.4. Calculate the number of sticks in a figure the perimeter of which is 105 cm.

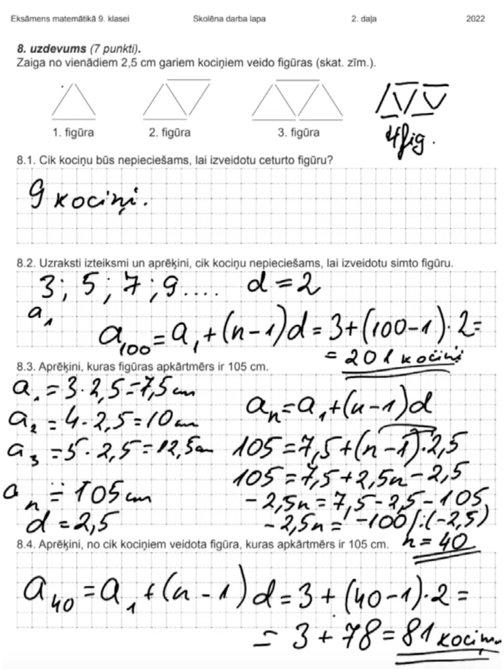
Remark. The formulation given by the authors of *Problem* 8 (see Figure 1) is not mathematically correct because, firstly, rule has not been given according to which the next shapes must be constructed and, secondly, it is bad that the ends of the sticks are not connected in the drawing. If the ends of the sticks have not been joined together, the question arises what is meant by the perimeter of the figure.

The fourth shape is probably assumed to be as in Figure 2. The next shapes could just as well be as in Figure 3.

Figure 2. Arrangement along the bar

Figure 3. Spiral arrangement

The situation with at least two different correct solutions is not always good for a pupil, because the teacher that corrects the solution of the pupil, might not consider other solutions that differ from the given one. For that teachers needs to be competent themselves to understand that other solution does not always mean that it is incorrect. On the Internet, you can find a solution that teachers are likely to show to their pupils as a good example of how to solve this problem, see Figure 4. This solution is neither the simplest nor the shortest.



Eksāmens matemātikā 9. klasei Skolēna darba lapa 2. daļa 2022

8. uzdevums *(7 punkti)*.
Zaiga no vienādiem 2,5 cm gariem kociņiem veido figūras (skat. zīm.).

1. figūra 2. figūra 3. figūra 4fig.

8.1. Cik kociņu būs nepieciešams, lai izveidotu ceturto figūru?

9 kociņi.

8.2. Uzraksti izteiksmi un aprēķini, cik kociņu nepieciešams, lai izveidotu simto figūru.

$3;\ 5;\ 7;\ 9\ldots \quad d=2$

a_1

$a_{100}=a_1+(n-1)d=3+(100-1)\cdot 2=$
$=201$ kociņi

8.3. Aprēķini, kuras figūras apkārtmērs ir 105 cm.

$a_1=3\cdot 2{,}5=7{,}5$ cm
$a_2=4\cdot 2{,}5=10$ cm
$a_3=5\cdot 2{,}5=12{,}5$ cm
$a_n=105$ cm
$d=2{,}5$

$a_n=a_1+(n-1)d$
$105=7{,}5+(n-1)\cdot 2{,}5$
$105=7{,}5+2{,}5n-2{,}5$
$-2{,}5n=7{,}5-2{,}5-105$
$-2{,}5n=-100 \,|:(-2{,}5)$
$n=40$

8.4. Aprēķini, no cik kociņiem veidota figūra, kuras apkārtmērs ir 105 cm.

$a_{40}=a_1+(n-1)d=3+(40-1)\cdot 2=$
$=3+78=81$ kociņi

Figure 4. Solution found on the YouTube (Jansons, 2022)

Here is a better solution.

Solution 8.2. Every next figure gains extra 2 sticks, this means that the number of the sticks in n-th figure can be calculated by formula

$$s(n) = 2n + 1 \Rightarrow s(100) = 201$$

Solution 8.3. The perimeter $p(n)$ formula is $p(n) = 2.5(2 + n)$. If

$$p(n) = 2.5(2 + n) = 105, \text{ then } 2.5n = 100, \text{ and } n = 40.$$

Solution 8.4. From the previous task we know that 40th shape has the perimeter equal to 105 cm, so $s(40) = 81$.

Some Problems about Magic Polygons

It is a familiar fact of mathematical instruction that a single good problem can awake a dormant mind more readily than highly polished lectures do.

Some instructive mathematics olympiad problems as well as topics for pupils' research papers to support, to encourage and to better equip teachers in their work with the gifted pupils are discussed. Several of these topics involve the study of geometric shapes – polyforms, which is not a part of the school mathematics curriculum. Polyforms are a rich source of problems, puzzles, and games, which are also quite suitable for workshops. Some such problems were presented at the 11th International Conference on Mathematical Creativity and Giftedness and have been published in proceedings of this conference (Bulina & Cibulis, 2019). In this paper some other problems from polyform topic, that are new and can be used in work with gifted pupils will be given and described.

Let us define some mathematical terms that will be used further in some of the given problems. A *polyomino* (*polyiamond*) is a plane shape that consists of unit squares (unit triangles) that are added to each other edge to edge. Here by *magic polygon*, we shall understand a squared or triangular polygon (a polyomino or polyiamond respectively) with all distinct whole sides: 1, 2, up to n (see Figure 5).

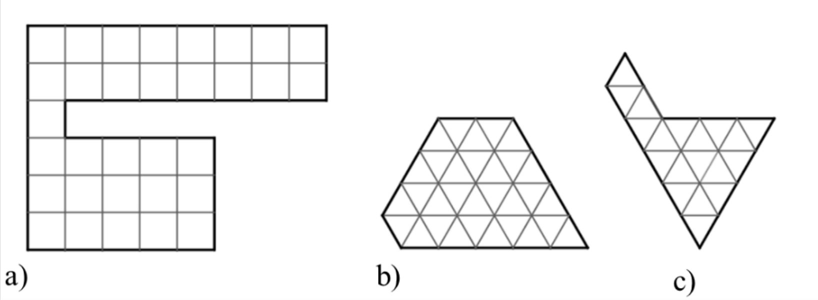


Figure 5. Magic polyomino and magic polyiamonds

If side lengths of a magic polygon are in the increasing order, it will be called *perfect* (see shape c) in Figure 5). In some literature (Dewdney, 1990; Sallows et al., 1991) perfect polygons on square grid are called *golygons*.

Let us look at some problems that deal with magic or perfect polygons.

Problem 1. Prove that there exists at least one magic polyiamond for each $n \geq 5$.

This problem can be easily solved with simple construction where magic polyiamond with $n = 2k + 1$ edges ($k > 2$) can be found by adding line segments to polyiamond with $n - 1$ sides from point X to Y with length n, 1 and $n + 1$ accordingly (see Figure 6). Similarly, you can construct magic polyiamonds with $n = 2k$, where $k \geq 3$.

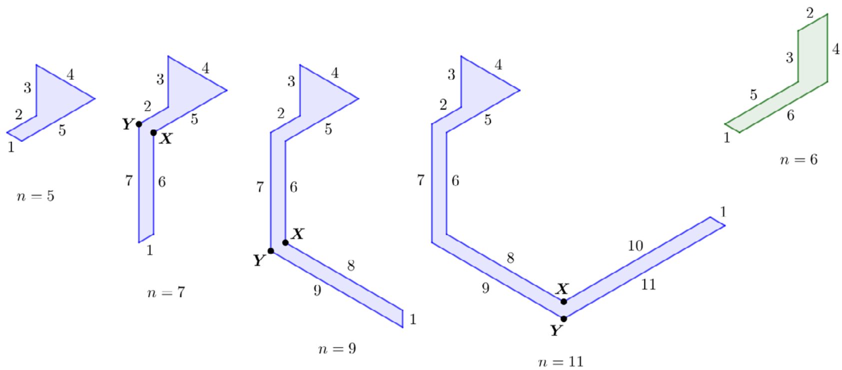


Figure 6. First four magic polyiamonds (found using given construction) and one magic hexagon

Finding such constructions is a good way to help gifted pupils to think outside the box and find some connection that might not be so easy to find at first. Problem 1 can be modified.

Problem 2. Prove that the construction in Figure 6 gives a magic polyiamond with the minimum area (polyiamond consists of the least number of triangles for fixed n).

Research topic 1. Find a construction that gives a magic polyomino with minimum area (polyomino consists of the least number of unit squares for fixed n).

Research topic 2. Find a construction that gives a magic polyomino with maximum area (polyomino consists of the greatest number of unit squares for fixed n).

In mathematics, as soon as new quantities are defined, there is a question about their existence. Here, the existence of perfect polyiamonds is a non-trivial question. This question can be divided in two separate problems (see Problem 3 and Problem 4).

Problem 3. Prove that for every even number $n \geq 6$ there exists a perfect polyiamond with n edges.

Solution to Problem 3. Since the number of edges is an even number $n = 2k$, here we use the idea of constructing only half of the broken line to start with, to obtain the polyiamond from two such halves. To implement the idea, we need to find broken lines with k edges which keep two fixed points (endpoints) after extending the edges. In mathematics, quantities that are preserved under a transformation are called *invariants*; the use of invariants is one of the big ideas of mathematics. After a little experimentation we can find some broken lines for which this invariance property holds, see Figure 7. Easy to check that if each segment length of the broken line (see A, B, C and D in Figure 7) is increased by 1 unit (dotted lines in Figure 7), the position of the broken lines endpoints (red and blue points in Figure 7) will remain the same.

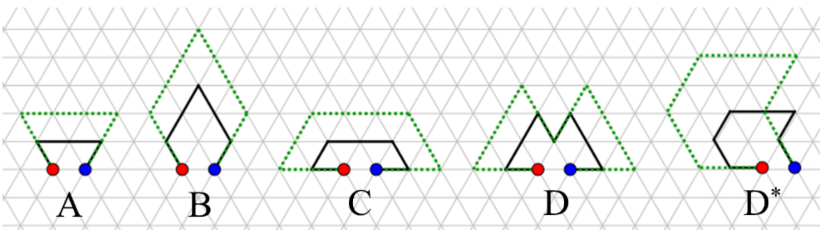


Figure 7. Broken lines with invariant position of its endpoints

The invariance property of the endpoints will also hold if the lengths of these broken line segments are different natural consecutive numbers (see A_1, B_1, C_1 and D_1 in Figure 8).

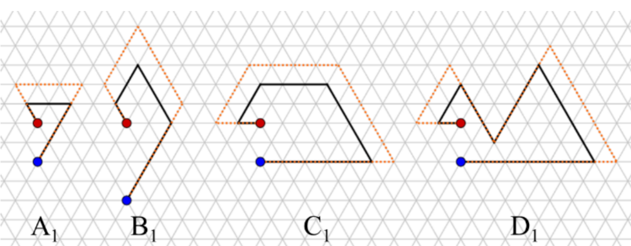


Figure 8 . Broken lines with consecutive edges

Let us use this result to prove that for each even n where $n > 4$ exists at least one perfect polyiamond. We separate 4 cases: $n = 6 + 8k$, $n = 8 + 8k$, $n = 10 + 8k$, $n = 12 + 8k$ (for all cases $k \geq 0$).

To find at least one perfect polyiamond for each $n = 6 + 8k$ where $k \in \mathbb{Z}_+$, we need to take two broken lines A from Figure 7 and add to each of them 4 edges (blue dotted lines in Figure 9) k times. For example, when $k = 1$ from broken lines A, B, C, D in Figure 7 we get broken lines with 7, 8, 9 and 10 segments (see Figure 9), while when $k = 2$ from broken lines A, B, C, D in Figure 7 we get broken lines with 11, 12, 13 and 14 segments (see Figure 10).

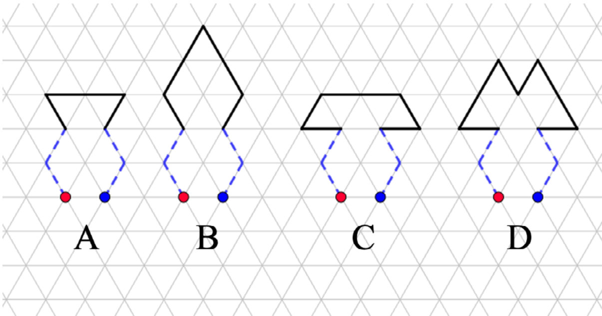


Figure 9. Broken lines with added dotted lines $k = 1$ times

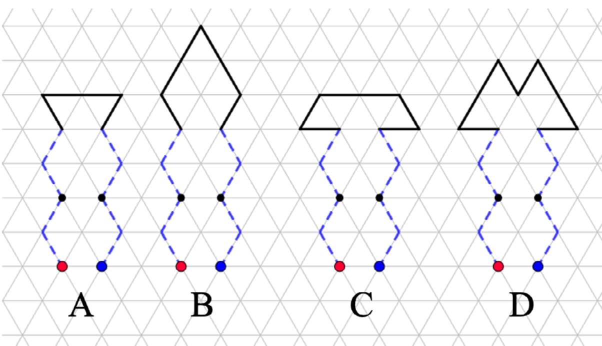


Figure 10. Broken lines with added dotted lines $k = 2$ times

To get magic polyiamond the two copies of a broken line of the same type, should be connected by endpoints of these broken lines. We then draw the resulting shape starting at one of the endpoints (connecting points) so that lengths of the edges starting at this point are consecutive numbers from 1 to n. For example, if $n = 14$, this construction gives a magic polyiamond shown in Figure 11.

Similarly, we can construct other perfect polyiamonds, except a perfect 10-gon. To show that there exists a perfect 12-gon, we need to take two copies of a broken line with 6 segments so the first and last segment both are not horizontal at the same time as it is for a broken line D in Figure 7.

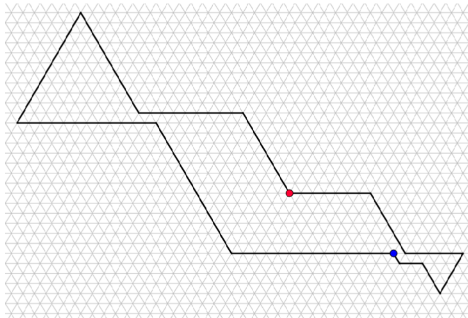


Figure 11. Perfect 14-gon

For example, using two copies of a broken line D* (see, Figure 7) we can construct a perfect 12-gon. From (Sallows, 1992) we also know that there exists four perfect 10-gons (polyiamonds). Moreover, none of these four perfect 10-gons can be obtained using construction given in Problem 3.

Problem 4. Prove that for every odd number $n \geq 5$ there exists a perfect polyiamond.

Problem 3 and Problem 4 are suitable topics that can be explored by gifted pupils with the aim of obtaining new constructions of perfect polyiamonds. Here we present one of the simplest constructions of perfect polyiamonds when the number of sides is $n = 8k + 3$.

We place the four edges, namely with the lengths $2k$, $4k + 1$, $6k + 2$ and $8k + 3$ horizontally, starting with the longest edge we assign orientation to all edges from 1 to n, as shown in Figure 12.

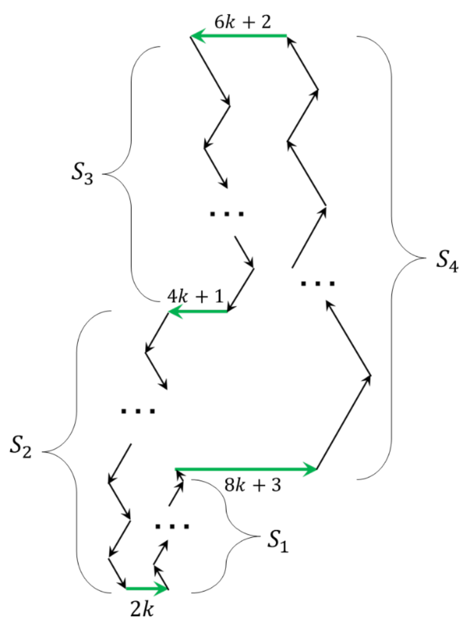


Figure 12. Perfect polyiamond with $n = 8k + 3$

Let us calculate the four sums S_1, S_2, S_3, and S_4 of edge lengths, see Figure 14. Applying the formula for the sum of the terms of an arithmetic progression, we obtain that

$$S_1 = 1 + 2 + \cdots + (2k - 1) = (2k - 1)k$$

$$S_2 = (2k + 1) + (2k + 2) + \cdots + 4k = (6k + 1)k$$

$$S_3 = (4k + 2) + (4k + 3) + \cdots + (6k + 1) = (10k + 3)k$$

$$S_4 = (6k + 3) + (6k + 4) + \cdots + (8k + 2) = (14k + 5)k.$$

From here, it is easy to see that

$$S_1 + S_4 = S_2 + S_3 \tag{1}$$

It means that the sum of edges with orientation ↗ and ↖ ("up") is equal to the sum of edges with orientation ↙ and ↘ ("down"). Similarly, the sum of edges with orientation → ("right") is equal to the sum of edges with orientation ← ("left")

$$2k + (8k + 3) = (4k + 1) + (6k + 2) \tag{2}$$

The mathematical interpretation of equalities (1) and (2) is that such construction does give a broken, closed line.

Remark. This construction was found in 2022 by Marta Rudzāte, a pupil of Form 12, now a student at the Faculty of Physics, Mathematics and Optometry, the University of Latvia. She has also been able to find perfect polyiamonds with $n = 8k + j$, $j = 1, 3, 5, 7$. Here we see that there are gifted pupils who can perform at Master's level.

Research topic 3. Show that there exists at least one perfect polyiamond for cases $n = 8k + 1$, $n = 8k + 5$ and $n = 8k + 7$. Is it possible to find a construction of perfect polyiamonds that gives $(2k + 1)$-polygons at once, i.e., without having to search separately four types of polygons?

Let us propose one more research topic to use in work with gifted pupils in their research works or contest papers. Some results could be modified to be used as problems in mathematics olympiads.

Research topic 4. What is the maximum (minimum) area of a polyomino having edges. What is the answer to the modified problem if additionally, we require polyomino to be perfect? Research these questions when considering polyiamonds instead of polyominoes.

Conclusions

Teacher shortages are a serious problem not only in Latvia, but also worldwide. This situation leads to a shortage of qualified teachers who can work not only on school subjects, but also at a higher level, supervising the research projects of gifted pupils.

The shortage of teachers has many negative effects, such as no competition, no contingent to work with at a higher level, and in this context, there are repeated assessments by education experts that the Latvian education system is catastrophically lacking excellence, and the overall level is very mediocre compared to other developed countries. A shortage of teachers creates a shortage of excellence, but excellence requires difficult tasks.

The compilers of the school task (see Figure 1) had to draw a triangle lattice instead of a square lattice, because the task requires drawing and counting triangles.

This article offers some research topics as well as mathematical ideas that may be useful for teachers working with their gifted pupils.

All the given research topics deals with polygons on a square or triangle grid. Exploring polygons of this type can easily be integrated into the new school curricula that emphasise the so-called big ideas in mathematics, such as analogy, abstraction, averaging, induction, symmetry, transformations, invariants, etc. Authors are always eager to learn about clever alternate solutions for challenging problems, and therefore would be pleased to hear from you should you have any.

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