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ABSTRACTS

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Abstracts

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BARTLETT CORRECTIONS FOR QUANTILE INFERENCE WITH EMPIRICAL LIKELIHOOD

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Empirical likelihood is a nonparametric approach to statistical inference which preserves many desirable properties of its parametric counterpart maximum likelihood. As in the parametric case the test procedures usually rely on the fact that statistic has an asymptotic chi square distribution. It is well known that in the case of maximum likelihood a scalar multiplier, known as Bartlett correction, can be derived which improves the accuracy of this approximation. Somewhat surprisingly it turns out that in many circumstances empirical likelihood can be Bartlett corrected as well. This is not only interesting from the theoretical point of view, since it implies that there is a deeper relation between parametric and empirical likelihoods, but it is also practically useful because we can derive statistical tests that are more accurate, especially when the sample sizes are small.

In a seminal paper DiCiccio et al. [1] showed that empirical likelihood can be Bartlett corrected when the parameter of interest can be expressed as a smooth function of means. Afterwards Bartlett corrections have been derived in many empirical likelihood related papers for many other models. In one such paper Chen and Hall [2] derived Bartlett corrections for quantile inference when kernel smoothing is applied to empirical likelihood. Undoubtedly this is a notable theoretical result, however from the simulation analysis the practical significance of the corrections seems questionable. It appears that when kernel smoothing is applied to empirical likelihood Bartlett corrections make almost no improvement. Hence the objective of this study is to investigate whether Bartlett corrections make a substantial improvement when confidence intervals for quantiles are constructed with smoothed empirical likelihood. In order to accomplish this the derivations made by Chen and Hall are reconstructed and an extensive simulation analysis is conducted.

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NUMERICAL ASPECTS OF MODULATION INSTABILITY

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Evolution of a nonlinear wave may result in modulation instability (MI): a spontaneous growth of the small amplitude modulations that transforms the initial wave in a sequence of pulses. MI was detected in numerous nonlinear wave systems, in optical fibers it is described by the generalized nonlinear Schrödinger equation (GNLSE) for the complex wave envelope $\psi(z, \tau)$

$$i\partial_z \psi + \hat{D}\psi + \gamma(|\psi|^2 - P_0)\psi = 0, \quad \text{where} \quad \hat{D} : e^{-i\nu\tau} \mapsto D(\nu)e^{-i\nu\tau}. \tag{1}$$

GNLSE "accepts" the initial temporal pulse shape $\psi(0, \tau)$ and "returns" the resulting pulse shape along the fiber, $\psi(z, \tau)$, where z is the propagation distance. The so-called *dispersion operator* $\hat{D}$ is usually represented by a polynomial $D(\nu)$ and is then a differential operator, which can be written as $\hat{D} = D(i\partial_\tau)$. In the long run, MI contributes to the formation of solitons, generation of wave turbulence, optical supercontinuum, formation of extreme waves, and so on, see [1].

This contribution deals with the numerical aspects of MI. Namely, the small wave modulations are so fundamentally simple, that it is possible to study them directly for the numerical schemes that solve GNLSE, such as the split-step Fourier method. A requirement that the numerical solution reproduces MI, while avoiding the nonphysical numerical instabilities, provides information on the applicability of the method. We derive the following statement

THEOREM. *If GNLSE* (1) *is solved numerically using the split-step Fourier method with the temporal resolution $\Delta\tau$, the evolution step h shall obey the inequality*

$$h < h_{max} = \frac{\pi}{\|D\| + 2|\gamma|P_0}, \quad \|D\| = \max_{|\omega_{b,r}| < \frac{\pi}{\Delta\tau}} \left| \frac{D(\omega_b) + D(\omega_r)}{2} - D\left(\frac{\omega_b + \omega_r}{2}\right) \right|. \tag{2}$$

Violation of the inequality (2) leads to slow excitation of the spurious resonant wave interactions, which may completely spoil the numerical solution for a realistic fiber length. The inequality is especially important to meet when an envelope equation has to be applied in a wide spectral window, e.g., because of the spectral broadening.

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COMPUTATION OF NUMERICALLY EXACT STATIONARY AND MOVING POLAROBREATHERS

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In this work, we develop a numerical algorithm based on nonlinear least squares with constraints to obtain numerically exact stationary and moving polarobreather solutions [1], i.e., discrete breathers carrying charge, electron or hole, in semiclassical crystal lattice models. For the time integration of the nonseparable canonical Hamiltonian system, we consider a recently proposed structure-preserving numerical method that exactly preserves the total charge probability [2]. The essential properties of stationary and moving polarobreathers are deduced from the lattice and charge variables spectra in the frequency-momentum space, where the wave-characterization of exact discrete breathers has been put forward in [3; 4] in one and two-dimensional crystal lattice models, respectively. Authors in [1] have extended the breather theory to polarobreathers in semiclassical models. The spectrum of approximate polarobreather allows for determining the necessary parameters for the proposed numerical algorithm to compute exact polarobreathers, i.e., the fundamental time or period, the charge energy shift value, and the spatial step for moving solutions. The spectra of exact polarobreathers become extremely simple and easy to interpret. Understanding the polarobreather spectrum is fundamental to interpreting possible signatures of localized nonlinear waves in the physical spectra of real crystals. With our proposed methodology, we also solve the problem that the charge frequency is not an observable, whereas the frequency of the charge probability, which is related to the discrete breather frequency, is an observable. Mathematical details of the methods, numerical results, and spectra of polarobreathers will be presented at the conference.

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ON THE LAPLACE TRANSFORM OF THE RIEMANN ZETA-FUNCTION

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Let $s = \sigma + it$ be a complex variable. The Laplace transform $\mathfrak{L}(s, f)$ of a function f is defined by

$$\mathfrak{L}(s, f) = \int_0^\infty f(x)e^{-sx}dx$$

provided that the integral exists for $\sigma > \sigma_0$ with some $\sigma_0 \in \mathbb{R}$. Denote by $\zeta(s)$ the Riemann zeta-function.

This report is a generalized case of [1], and is devoted to the Laplace transform

$$\mathfrak{L}\left(s, |\zeta|^{2k}\right) = \int_0^\infty \left|\zeta\left(\frac{1}{2} + ix\right)\right|^{2k} e^{-sx}dx,$$

where $k \in \mathbb{N}$. In [1], the case $k = 2$ has been considered. In the report, we obtain explicit formulae involving some elementary functions for $\mathfrak{L}\left(s, |\zeta|^{2k}\right)$ with arbitrary $k \in \mathbb{N}$.

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ON PARAMETERIZED METRIC AND ITS APPLICATIONS IN COMBINATORICS ON WORDS

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In the literature there are several kinds of metrics on the set X of infinite words. The first one is defined as follows, see e.g. [1]. Let $x, y \in X$ be infinite words. Then

$$\rho(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 2^{-n} & \text{otherwise where } n = \min\{i : x_i \neq y_i\}, \end{cases}$$

is a metric on the set of infinite words, where $\rho : X \times X \to [0; 1]$.

Another known definition of a metric on the set of infinite words is introduced as follows, see e.g. [2]. Let $x, y \in X$ be infinite words, and let for a given $i \in \mathbb{N}$ the number χ_i be defined by:

$$\chi_i(x,y) = \begin{cases} 0 & \text{if } x_i = y_i \text{ where } i \text{ is the } i\text{-th coordinate of the word,} \\ 1 & \text{if } x_i \neq y_i \text{ where } i \text{ is the } i\text{-th coordinate of the word.} \end{cases}$$

Now let

$$\sigma(x,y) = \sum_{i=0}^{\infty} \frac{1}{2^i} \chi_i(x,y).$$

Then $\sigma : X \times X \to [0, 2]$ is a metric (actually ultrametric) on the set of all infinite words. We give some counterexamples for both these metrics, where results contradict our intuition.

We suggest to use parameterized metric, see [3], and define a parameterized metric, which is suitable for comparing distance between two infinite words (also comparing their prefixes) defining:

$$P(x,y,t) = \frac{2}{\lceil t \rceil \cdot \lceil t + 1 \rceil} \sum_{i=1}^{\lceil t \rceil} \chi_i(x,y) \left(\lceil t + 1 \rceil - i \right).$$

We give some results for counterexamples, general cases and particularly for periodic words.

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AMPLITUDE EVOLUTION EQUATIONS FOR THE ANALYSIS OF NON-ISOTHERMAL INCOMPRESSIBLE FLOWS

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The paper deals with weakly nonlinear approach to stability analysis of non-isothermal flows under the Boussinesq approximation. We consider a tall vertical channel bounded by two infinitely long parallel planes. The channel is assumed to be closed at infinity. Steady convective flow in a vertical direction can be generated by two factors: (a) different constant wall temperatures and/or (b) internal heat sources induced in the fluid. We assume that there exists a steady base flow in the vertical direction depending on the transverse coordinate.

Imposing small perturbations on the flow and linearizing the governing equations in the neighborhood of the base flow we obtain linear stability equations which are coupled with boundary conditions. The corresponding boundary value problem is solved numerically and the critical values of the Grashof number (the parameter characterizing the flow) are found. Linear stability theory gives the conditions when the given steady flow becomes unstable but it cannot be used to analyze the development of instability above the threshold. In this case one can use weakly nonlinear theory. It is assumed that the Grasshof number is slightly above the critical value so that the base flow is unstable but the growth rate of a perturbation is small.

Method of multiple scales [1], [2] is used in the paper to develop amplitude evolution equation for the most unstable mode. It is shown that for the case of planar perturbations the evolution equation is the complex Ginzburg-Landau equation. The coefficients of the equation are expressed in terms of integrals containing solutions of several boundary value problems.

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THE INVESTIGATION OF DATA TEMPORAL DELAYS IMPACT ON CLASSIFICATION PERFORMANCE FOR CHURN PREDICTION IN TELECOMMUNICATIONS

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The field of Machine Learning (ML) is focused on developing models based on standardized benchmarks, which are used to compare performance among developers. The benchmarks rely on datasets that are either based on real data or synthesized according to observations from real data. ML models often aim to predict outcomes through either classification or regression. However, there are important questions that are often overlooked, which can result in discrepancies between a model's performance during research and its performance in real-world applications. In this talk, we will address some of these key issues, including:

- The behavior patterns of observed objects can change over time, which can cause delays between prediction and training intervals and affect the accuracy of results. While this delay cannot be avoided, it must be considered when optimizing the practical outcome of the technology, as demonstrated in [1].

- The classic approach of splitting a dataset into training, validation, and testing parts does not allow for precise estimation of a model's performance in a real-world scenario. Therefore, an additional set of data must be included, which we will refer to as "true testing data". This data must be qualitatively different from the training/validation data and should consist of information from the future that can simulate and evaluate the model's application with a delay.

- The effect of delay in training and prediction can vary depending on the specific time window, so it's important to observe different time windows.

Throughout this discussion, we will provide an illustrative case study of churn prediction in telecommunications. In this context, a churner is defined as a user who has stopped using a specific service, such as telecommunication services from a specific operator. While the most common definition of a churner in telecommunications is a client who has not generated any revenue for three months, alternative definitions may lead to better prediction or retention results, as noted in [1].

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ONE NEURON MODEL AS CHAOTIC ATTRACTOR

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We study the following non-autonomous piecewise linear difference equation

$$x_{n+1} = \beta_n x_n - g(x_n), \quad n = 0, 1, 2, ..., \tag{1}$$

where $(\beta_n)_{n=0}^{\infty}$ is a period two or three sequence

$$\beta_n = \left\{ \begin{array}{ll} \beta_0, & \text{if } n = 2k, \\ \beta_1, & \text{if } n = 2k+1, \end{array} \right. \quad \text{or} \quad \beta_n = \left\{ \begin{array}{ll} \beta_0, & \text{if } n = 3k, \\ \beta_1, & \text{if } n = 3k+1, \\ \beta_2, & \text{if } n = 3k+2, \end{array} \right. \quad k = 0, 1, 2, ...,$$

and g is the piecewise constant function with McCulloch-Pitts nonlinearity

$$g(x) = \left\{ \begin{array}{ll} 1, & x \geq 0, \\ -1, & x < 0. \end{array} \right. \tag{2}$$

In [1] a difference equation $x_{n+1} = \beta x_n - g(x_n), \quad n = 0, 1, 2, ...,$ with (2) was analyzed as a single neuron model where $\beta > 0$ is an internal decay rate and g is a signal function. In [2; 3; 4] we have been studied this model where $(\beta_n)_{n=0}^{\infty}$ is a period two and three sequences. We investigated the periodic behavior and stability of the solutions depending on the size of $(\beta_n)_{n=0}^{\infty}$.

Now we show that for certain values of the coefficients there exists an attracting interval for which the model is chaotic. On the other hand, if the initial value is chosen outside the mentioned attracting interval, then the solution of the difference equation either increases to positive infinity or decreases to negative infinity. The properties of chaos (sensitivity to initial conditions) can be used in random number generation as well as in cryptography.

The results are obtained in collaboration with M.A. Radin, Rochester Institute of Technology, U.S.A.

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EFFICIENT NUMERICAL ALGORITHMS FOR PARTIALLY DIMENSION REDUCED NONLOCAL HEAT CONDUCTION PROBLEMS

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We construct and investigate efficient parallel finite volume schemes for solution of partially dimension reduced nonlocal heat conduction problems. Let us restrict to 2D space domains. In the case of classical diffusion operators the following hybrid dimension PDE model is defined in [1]:

$$\frac{\partial U}{\partial t} + AU = 0, \quad (x, y, t) \in (0, \delta) \cup (X - \delta, X) \times (0, Y) \times (0, T], \tag{1}$$

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \quad (x, t) \in (\delta, X - \delta) \times (0, T], \tag{2}$$

$$U(0, y, t) = g_1(y, t), \quad U(X, y, t) = g_2(y, t), \quad (y, t) \in [0, Y] \times (0, T],$$

$$\frac{\partial U}{\partial y}(x, 0, t) = 0, \quad \frac{\partial U}{\partial y}(x, Y, t) = 0, \quad (x, t) \in (0, \delta) \cup (X - \delta, X) \times (0, T].$$

The following two pairs of conjugation conditions are valid at the truncation lines:

$$U\big|_{x_\delta - 0} = U\big|_{x = \delta + 0}, \quad U\big|_{x = X - \delta - 0} = U\big|_{x = X - \delta + 0},$$

$$\frac{\partial S(U)}{\partial x}\bigg|_{x_\delta - 0} = \frac{\partial U}{\partial x}\bigg|_{x = \delta + 0}, \quad \frac{\partial U}{\partial x}\bigg|_{x = X - \delta - 0} = \frac{\partial S(U)}{\partial x}\bigg|_{x = X - \delta + 0}. \tag{3}$$

Our main aim is to generalize this technique for the nonlocal fractional power diffusion operators A^α, where $0 < \alpha < 1$. The spectral definition of such operators is used in our analysis.

Numerical finite volume schemes for solution of parabolic problems with fractional power elliptic operators are investigated in [2]. Here a special theoretical framework is proposed to split the full convergence analysis into three separate parts. For any new problem it is sufficient to analyze only specific parts of this problem.

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1D CONVOLUTIONAL NEURAL NETWORKS FOR SOLUTION OF THE BIOSENSOR INVERSE PROBLEM

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Biosensors are devices for the detection and analysis of chemical compounds based on biochemical processes. Since real-world experiments involving biosensors are expensive and time consuming, mathematical modelling is widely used in their development [1]. Our main interest is in studying the inverse biosensor problem – given a biosensor signal for an unknown sample, determine its composition. This requires solving the forward model equations for signals, whose composition is known, to establish an inverse dependence. Once this is done, unknown samples can be analyzed. But, unless we have a biosensor for a single substrate, this process is less straightforward, since for multiple substrates this problem is ill-posed [2]. This may lead to the deterioration of biosensor precision, especially when the biosensor response is under the effect of noise [3].

One promising method to alleviate the ill-posedness of such problems is to use neural networks, which have successfully been applied to solve the inverse biosensor problem [4]. Here, the authors use the feedforward neural network architecture, however, this raises the question whether such a simple architecture is optimal for this particular use case when the biosensor signals are corrupted by noise. In this presentation, we propose to use one-dimensional convolutional neural networks for the biosensor problem. This architecture is employed in spectroscopy [5], where obtained spectra are often noisy, due to its ability to remove noise [6]. We present our findings on biosensor precision for noisy signals, and the effect of measurement domain shrinkage for this architecture.

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A STUDY OF THERMAL DECOMPOSITION OF POROUS BIOMASS

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This paper deals with theory and simulations for mathematical model of biomass thermal decomposition. Objective of the study is to simulate pyrolysis of straw, wood and peat mixtures with different microwave pretreatments. Biomass thermal decomposes as a volatile part and carbon part which reacts with oxygen creating additional heat. Biomass is viewed as a porous medium. Chemical reactions are modeled according to Arrhenius kinetics. Biomass is described by components lignin, cellulose and hemicellulose for the purpose to model the thermal decomposition difference between straw, wood and peat. Gases in this model is viewed like an ideal gas which covered by the Darcy law and mass balance equation [1]. Porosity changes within process of biomass thermal decomposition. Numerical solutions were found using upwind and finite volume methods in program MatLab.

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UNIVERSALITY OF CERTAIN COMPOSITIONS

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Let $s = \sigma + it$ be a complex variable, and $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ a periodic sequence of complex numbers. The periodic zeta-function $\zeta(s; \mathfrak{a})$, for $\sigma > 1$, is given by the Dirichlet series $\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} a_m m^{-s}$, and by analytic continuation elsewhere. If the coefficients are multiplicative, then the function $\zeta(s; \mathfrak{a})$ is universal [1], i. e., its shifts $\zeta(s + i\tau; \mathfrak{a})$ approximate a class of analytic functions defined on the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. In [2] and [3] joint universality theorems on approximation of collection of analytic functions by shifts $(\zeta(s + i\gamma_1(\tau); \mathfrak{a}_1), \ldots, \zeta(s + i\gamma_r(\tau); \mathfrak{a}_r))$ and $(\zeta(s + ih_1\gamma_1; \mathfrak{a}_1), \ldots, \zeta(s + ih_r\gamma_r; \mathfrak{a}_r))$, respectively, were proved. Here $\gamma_1(\tau), \ldots, \gamma_r(\tau)$ are certain differentiable functions, and $\{\gamma_k : k \in \mathbb{N}\}$ is a sequence of positive imaginary parts of non-trivial zeros of the Riemann zeta-function.

Let $H(D)$ be the space of analytic on D functions. In the report, we consider the universality of the functions $F(\zeta(s; \mathfrak{a}_1), \ldots, \zeta(s; \mathfrak{a}_r))$, where $F : H^r(D) \to H(D)$ is a certain operator. We give one example. Denote by $\mathcal{K}$ the class of compact sets of D, and by $H(K)$, $K \in \mathcal{K}$, the class of continuous functions on K that are analytic in the interior of K, and let $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$.

THEOREM 1. *Suppose that the sequences $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are multiplicative, $h_1, \ldots, h_r$ are positive algebraic numbers linearly independent over $\mathbb{Q}$, and estimate $\sum_{(\gamma_k, \gamma_l): |\gamma_k - \gamma_l| < 1/\log T} 1 \ll T \log T$, $T \to \infty$, is valid. Let $F : H^r(D) \to H(D)$ be a continuous operator, such that $\{g \in H(D) : g(s) \neq c \in \mathbb{C}\} \subset F(S^r)$. Let $K \in \mathcal{K}$, $f(s) \in H(K)$ and $f(s) \neq c$ on K. Then, for every $\varepsilon > 0$,*

$$\liminf_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s \in K} |F(\zeta(s + ih_1\gamma_k; \mathfrak{a}_1), \ldots \zeta(s + ih_r\gamma_k; \mathfrak{a}_r)) - f(s)| < \varepsilon \right\} > 0.$$

For example, we can take $F(g_1, \ldots, g_r) = b_1 g_1 + \cdots + b_r g_r$, $g_1, \ldots, g_r \in H(D)$, $b_1, \ldots, b_r \in \mathbb{C} \setminus \{0\}$.

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STOCHASTIC MODELING OF ANIMAL POPULATION DYNAMICS

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The most popular classical mathematical model for biological pest control may be given as a system of ordinary differential equations (see, for example, in A. D. Bazykin at all [1] and Yu. M. Svirezhef and D. I. Logofet [2] system of the equations (1)):

$$\begin{cases} \frac{dx}{dt} = xf(x) - g(x,y), \\ \frac{dy}{dt} = yh(y) + m(x,y), \end{cases} \tag{1}$$

where x is a density of a pest (or prey) population and y is a density of a predator population, functions $f(x)$ and $h(y)$ define the relative growth rates of populations without contacts, functions $g(x,y)$ and $m(x,y)$ are changes of populations, growth rates caused by a cooperative effect, called functional responses. In reality, even through all functions of the right part in (1) are derived in compliance with biological laws, the parameters of these functions are random and should be estimated by collecting and analyzing environmental data. Therefore, several papers propose and analyze mathematical models for interacting populations in a form of stochastic Ito differential equations (see, for example, the book of J. Murray [3] and references there):

$$\begin{cases} dx(t) = \big[x(t)f\big(x(t)\big) - g\big(x(t),y(t)\big)\big]dt + \sigma_1\big(x(t),y(t)\big)dw_1(t), \\ dy(t) = \big[y(t)h\big(y(t)\big) + m\big(x(t),y(t)\big)\big]dt + \sigma_2\big(x(t),y(t)\big)dw_2(t), \end{cases} \tag{2}$$

where $w_1(t)$ and $w_2(t)$ are correlated Wiener processes given on probability space $(\Omega, \mathfrak{F}, \mathbb{P})$.

The paper deals with population mathematical models given in a form of fast oscillating stochastic impulsive differential equations. The proposed approximative algorithm for quantitative analysis of population dynamics consists of two steps. First, we construct an ordinary differential equation for the mean value of population growth and analyse the average asymptotic population behaviour. Then, applying diffusion approximation procedure, we derive the stochastic Ito differential equation for random deviations on the above average motion and analyse probabilistic characteristics of possible stationary population state. The algorithm is applied to the model of biological pest control of Holling type II.

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AGGREGATION BASED FUZZY RELATIONS IN ANTI-MONEY LAUNDERING MONITORING ALGORITHM

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Anti-Money Laundering (AML) is a set of laws and regulations designed to prevent the illegal concealment of money obtained through criminal activity. Anti-Money Laundering typically involves analyzing large amounts of financial transaction data to identify patterns or behaviors that may indicate money laundering or other illicit financial activities. Various mathematical algorithms and techniques, mostly machine learning algorithms, can be used in AML to help detect and prevent such activities. However, in AML, the data sets are often highly imbalanced, with a small proportion of transactions being suspicious or fraudulent. This presents a challenge for machine learning algorithms, as they rely on having a large amount of data to learn from and make accurate predictions. When the number of bad transactions is small, what usually happens in real life, it becomes difficult for machine learning algorithms to identify patterns or features that distinguish them from the vast majority of legitimate transactions. The other tools which is used for ALM is rule-based algorithms – a type of approach that uses pre-defined rules to detect potentially suspicious financial transactions. These rules are based on specific criteria that are known for decition maker. Rule-based algorithms can also be implemented using fuzzy techniques and aggregation functions [2]. Overall, rule-based algorithms can be a useful tool in an AML program, but they should be used in conjunction with other methods such as machine learning and expert analysis to achieve the best results.

In our work we propose an alernative approach to existing ones, where rule-based algorithm is presenting by an aggregation function [1] and the degree of similarity to a suspicious transaction is determined by a fuzzy equivalence relation [3]. Thus, we build a aggregation based fuzzy equivalence relation, which is going to allow us identify the degree to which a transaction is illicit one.

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LINEAR STABILITY OF A COMBINED CONVECTIVE FLOW IN AN ANNULUS

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We consider convective flow in a tall vertical annulus. The annulus is assumed to be closed so that the total fluid flux through the cross-section of the channel is equal to zero. Steady convective flow in the vertical direction is generated due to two factors: (a) different constant temperatures of the walls and (b) internal heat generation due to chemical reaction that takes place in the fluid [1]. The problem is characterized by three dimensionless parameters: the Grasshof number Gr, the Frank-Kamenetskii parameter F characterizing heat release due to chemical reaction and the Prandtl number Pr.

The boundary value problem for steady state temperature distribution is nonlinear. Bifurcation analysis performed in the paper (similarly to the approach used [2]) shows that for each $R = R_1/R_2$, where R_1 and R_2 are the inner and outer radii of the annulus, respectively, there are two solutions in the interval $0 < F < F^*$, where F^* depends on R. The solution with the smallest norm is physically realizable and should be selected for linear stability analysis.

Linear stability problem is solved numerically for two cases considered in the paper: Case 1 (different constant wall temperatures) and Case 2 (zero wall temperatures). In Case 1, multiple minima on the marginal stability curves are obtained as the Prandtl number increases. The global minimum is selected among two or three local minima associated with different wave numbers and depends on the particular value of the Prandtl number. As a result, finite jumps in the value of the wave number can be observed for different Pr leading to the changes in the form of the most unstable perturbation. The critical Grashof numbers for Case 1 do not change much until the transition to the mode with different wave number takes place, after that the critical Gr decreases rapidly. Calculations show that for Case 2 marginal stability curves have one minimum and the critical Grashof numbers decrease monotonically as the Prandtl number grows. In both cases (Case 1 and Case 2) the increase in F leads to less stable flow.

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MULTIPLICITY RESULTS FOR BOUNDARY VALUE PROBLEMS WITH THE ARRHENIUS NONLINEARITY

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Various models of heat conduction uses the Arrhenius nonlinearity

$$f(T) = e^{-\frac{\theta}{T}}, \quad \theta > 0,$$

representing the temperature distribution.

We consider the following boundary value problems:

$$T'' + \frac{T'}{r} + F\,e^{-\frac{\theta}{T}} = 0, \qquad\qquad T(R) = 0 = T(1); \qquad\qquad (1)$$

$$T'' + \frac{T'}{r} + F\,e^{-\frac{\theta}{T}} = 0, \qquad\qquad T(R) = T_f = T(1); \qquad\qquad (2)$$

$$Q'' + \frac{Q'}{r} + Fe^{-\frac{\theta}{Q+T_f}} = 0, \qquad\qquad Q(R) = 0 = Q(1), \qquad\qquad (3)$$

where θ, F, T_f are positive numbers and $R \in (0, 1)$.

We are interested in positive solutions of problems (1)-(3).

Bifurcation analysis of these problems with respect to the parameters θ, F, T_f, and R allows us to obtain some results on the existence and number of positive solutions of (1)-(3).

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TAMING COMPLEXITY WITH OPTIMAL CONTROL: FROM ECOLOGY TO EPIDEMIOLOGY

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Most real-life problems are characterized by their inherent complexity. Ecological systems may undergo abrupt changes in their structure and functioning, which are typically referred to as regime switches. Once the regime shift happened, the system cannot return to its previous state, until another threshold is crossed. This results in the occurrence of multi-stability and hysteresis-like behavior, which is observed in the populations of insects and further natural phenomena, see [3]. When we turn to the epidemiological applications, their inherent complexity is caused not only by the highly non-linear dynamics, but also by a complicated interplay of structural and budgetary restrictions. This complexity substantially restricts our ability to determine the strategies aimed at combatting the spread of the disease.

In this talk, we will argue that optimal control theory offers a comprehensive framework for dealing with such problems. Due to its flexibility and a wide range of developed methods, it can address a broad spectrum of real problems. To illustrate this thesis, we will consider the application of hybrid optimal control, particularly the hybrid Pontryagin maximum principle, [1], to the problem of pollution control with periodic regime shifts, [4]. Another application comes from the realm of numerical optimal control and demonstrates the computation of the optimal treatment and prophylaxis investment profiles for the problem of reducing the incidence of HIV while obeying complex budgetary restrictions, [2].

The talk will be concluded by a brief overview of open problems and future directions of research.

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ON HEURISTIC PARAMETER CHOICE FOR THE CLASS OF REGULARIZATION METHODS

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Let H and F be Hilbert spaces and $A \in L(H, F)$. We consider the ill-posed operator equation

$$Au = f, \quad f \in \mathcal{R}(A) \neq \overline{\mathcal{R}(A)}. \tag{1}$$

The kernel $N(A)$ may be non-trivial. Instead of an exact right-hand side f_* we have only an approximation $f \in F$. For the regularization of problem (1) we consider the following class of regularization methods:

$$u_r = (I - A^* A g_r(A^* A)) u_0 + g_r(A^* A) A^* f.$$

Here u_0 is the initial approximation, r is the regularization parameter, I is the identity operator and the generating function $g_r(\lambda)$ satisfies for $r \geq 0$ the conditions

$$\sup_{0 \leq \lambda \leq \|A^* A\|} \left| g_r(\lambda) \right| \leq \gamma r, \qquad \sup_{0 \leq \lambda \leq \|A^* A\|} \lambda^p \left| 1 - \lambda g_r(\lambda) \right| \leq \gamma_p r^{-p}, \quad 0 \leq p \leq p_0.$$

Examples of methods of this class are (iterated) Tikhonov method, Landweber iteration method, implicit iteration method, method of asymptotical regularization, the truncated singular value decomposition methods etc.

If the noise level of data is unknown, for the choice of the regularization parameter r heuristic rule is needed. We propose to choose r from the set L_{min} of the local minimum points of the quasioptimality criterion function

$$\psi_Q(r) = r \| A^* (I - AA^* g_r(AA^*))^{\frac{2}{p_0}} (Au_r - f) \|$$

on the set of parameters $\Omega = \left\{ r_j : r_j = q r_{j-1}, j = 1, 2, ..., M, q > 1 \right\}$. Then the following error estimates hold:
a)

$$\min_{r \in L_{min}} \| u_r - u_* \| \leq C \min_{r_0 \leq r \leq r_M} \left\{ \| u_r^+ - u_* \| + \| u_r - u_r^+ \| \right\}.$$

Here u_* and u_r^+ are the exact and regularized solutions of equation $Au = f_*$ and the constant $C \leq c_q \ln(r_M / r_0)$ can be computed for each individual problem $Au = f$.
b) Let $u_* = (A^* A)^{p/2} v$, $\|v\| \leq \rho$. If $r_0 = 1$, $r_M = c \| f - f_* \|^{-2}$, $c = (2 \| u_* \|)^2$, then

$$\min_{r \in L_{min}} \| u_r - u_* \| \leq c_{p,q} \rho^{1/(p+1)} \left| \ln \| f - f_* \| \right| \| f - f_* \|^{p/(p+1)}, 0 < p \leq 2p_0.$$

We consider also some algorithms for parameter choice from the set L_{min}.

ON THE MISHOU THEOREM FOR PERIODIC ZETA-FUNCTIONS

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Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ and $\mathfrak{b} = \{b_m : m \in \mathbb{N}_0\}$ be two periodic sequences of complex numbers, $0 < \alpha \leqslant 1$ be a fixed number, and $s = \sigma + it$. The periodic and periodic Hurwitz zeta-functions are defined, for $\sigma > 1$, by the series $\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}$, $\zeta(s, \alpha; \mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s}$, and have meromorphic continuations to the whole complex plane.

In this talk, we will consider the joint approximation of analytic functions by shifts of some absolutely convergent Dirichlet series connected to the functions $\zeta(s; \mathfrak{a})$ and $\zeta(s, \alpha; \mathfrak{b})$. Our result extends the well-known Mishou theorem for the Riemann and Hurwitz zeta-functions [1].

Denote by $v_u(m) = \exp\{-(m/u)^{\theta}\}$, $m \in \mathbb{N}$, and $v_u(m, \alpha) = \exp\{-((m+\alpha)/u)^{\theta}\}$, $m \in \mathbb{N}_0$, with $u > 0$ and fixed $\theta > \frac{1}{2}$, and define the series $\zeta_u(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m v_u(m)}{m^s}$ and $\zeta_u(s, \alpha; \mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m v_u(m,\alpha)}{(m+\alpha)^s}$, which are absolutely convergent for $\sigma > 1$. We will consider the approximation of analytic functions by shifts $(\zeta_{u_T}(s + i\tau; \mathfrak{a}), \zeta_{u_T}(s + i\tau, \alpha; \mathfrak{b}))$, where $u_T \to \infty$ as $T \to \infty$. Now, define two tori $\Omega_1 = \prod_p \gamma_p$ and $\Omega_2 = \prod_{m \in \mathbb{N}_0} \gamma_m$, where γ_p and γ_m are the unit circles (for all $p \in \mathbb{P}$ and $m \in \mathbb{N}_0$), and let $\Omega = \Omega_1 \times \Omega_2$. Thus, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined. Denote by $\omega_1(p)$ the pth component of an element $\omega_1 \in \Omega_1$, and by $\omega_2(m)$ the mth component of an element $\omega_2 \in \Omega_2$, and construct two random elements $\zeta(s, \omega_1; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega_1(m)}{m^s}$ and $\zeta(s, \alpha, \omega_2; \mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m \omega_2(m)}{(m+\alpha)^s}$, where the function $\omega_1(m)$ is the extension of $\omega_1(p)$ to the set $\mathbb{N}$. Note that the latter series are uniformly convergent on compact subsets of the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ for almost all ω_1 and ω_2. Moreover, let $\mathcal{K}$ be the class of compact subsets of D with connected complements, $H(K)$ (with $K \in \mathcal{K}$) the class of continuous functions on K that are analytic in the interior of K, and $H_0(K)$ is the subclass of $H(K)$ consisting of non-vanishing functions.

THEOREM 1. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, α is transcendental, $u_T \to \infty$ and $u_T \ll T^2$ as $T \to \infty$. Let $K_1, K_2 \in \mathcal{K}$ and $f_1(s) \in H_0(K), f_2(s) \in H(K)$. Then the limit*

$$\lim_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{s \in K_1} |\zeta_{u_T}(s + i\tau; \mathfrak{a}) - f_1(s)| < \varepsilon_1, \sup_{s \in K_2} |\zeta_{u_T}(s + i\tau, \alpha; \mathfrak{b}) - f_2(s)| < \varepsilon_2 \right\}$$

$$= m_H \left\{ (\omega_1, \omega_2) \in \Omega : \sup_{s \in K_1} |\zeta(s, \omega_1; \mathfrak{a}) - f_1(s)| < \varepsilon_1, \sup_{s \in K_2} |\zeta(s, \alpha, \omega_2; \mathfrak{b}) - f_2(s)| < \varepsilon_2 \right\} > 0$$

exists for all but at most countably many $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.

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NAVIER-STOKES EQUATION IN A THIN TUBE STRUCTURE MOTIVATED BY HEMODYNAMICS

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The steady-state Navier-Stokes equation in a thin tube structure B_ε, with the inflow-outflow boundary conditions for the Bernoulli pressure, is considered (see (1))

$$
\begin{cases}
-\nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0, & x \in B_\varepsilon, \\
\operatorname{div} \mathbf{v} = 0, & x \in B_\varepsilon, \\
\mathbf{v} = 0, & x \in \partial B_\varepsilon \setminus \cup_{j=N_1+1}^{N} \gamma_\varepsilon^j, \\
\mathbf{v}_{\boldsymbol{\tau}} = 0, & x \in \gamma_\varepsilon^j, \\
-\nu \partial_n (\mathbf{v} \cdot \mathbf{n}) + \left(p + \dfrac{1}{2}|\mathbf{v}|^2 \right) = c_j/\varepsilon^2, & x \in \gamma_\varepsilon^j, \ j = N_1+1, ..., N,
\end{cases}
\tag{1}
$$

where $\mathbf{v}_{\boldsymbol{\tau}} = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n}$ is the tangential component of the vector $\mathbf{v}$, $\partial_n h = \nabla h \cdot \mathbf{n}$ is the normal derivative of h, c_j are some constants. The existence and uniqueness of the weak solution taking into account the domain dependence on the small parameter were proved. Also, the asymptotic expansion of a weak solution of the stationary Navier-Stokes equations in the whole thin tube structure with the given Bernoulli boundary conditions for the inflows and outflows was constructed. The constructed asymptotic expansion allowed reducing the computational costs. Indeed, full-dimensional computations are only needed in small neighborhoods of the junction of tubes, while in the main part of the domain, the computations are one-dimensional. Joint work with Grigory Panassenko and Konstantinas Pileckas.

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REMARKS ON SIMULTANEOUS APPROXIMATION BY THE CLASSES OF ZETA-FUNCTIONS

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The universality property in the Voronin sense of the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, says roughly speaking that any analytic non-vanishing function can be approximated uniformly on any compact subsets of the strip $\{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ by shifts of $\zeta(s)$. The mixed joint universality gives a possibility to connect into one tuple two different types of zeta-functions (one of those zeta-functions has the Euler product expression over primes and the other does not) and to study an analogous problem on the approximation. In other words, we can show that any two target functions can be approximated simultaneously by the suitable vertical shifts of two different type zeta- or L-functions and that the set of such shifts has a positive lower density.

The rather general result can be shown for the broad classes of zeta-functions, particularly, if an approximating tuple is composed of the Matsumoto zeta-functions' class and the periodic Hurwitz zeta function.

In the talk, we will present three cases of the mixed joint discrete universality property of abovementioned functions. These results are obtained by the author, Kohji Matsumoto (Nagoya University, Japan) and Łukasz Pańkowski (A. Mickiewicz University, Poland) (see [1], [2], [3], [4]).

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ON INDUCED BIPOLAR OWA OPERATORS

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Ordered weighted averages (OWA) introduced by Yager [5] have shown their usefulness in multicriteria decision-making, and also in other fields where ranking is important. Later, OWA were generalized to induced OWA (IOWA) by Yager [6]. Induced means that the ranking is done by an inducing vector rather than by the vector itself to be aggregated. Recently, several other generalizations of OWA were introduced and studied. Important for our purposes will be also bipolar ordered weighted averages (BIOWA) recently introduced by Mesiar et al. in [3,4] that are based on the Choquet-integral with respect to bi-capacities (see [1,2]).

The main aim of this presentation is to introduce and investigate a generalization of BIOWA operators, in particular, induced bipolar ordered weighted averages.

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MATHEMATICAL MODELLING ELECTRICALLY DRIVEN FREE SHEAR FLOWS IN A DUCT UNDER UNIFORM MAGNETIC FIELD

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We consider a mathematical model of two-dimensional electrically driven laminar free shear flows in a straight duct under action of an applied uniform homogeneous magnetic field. The mathematical approach is based on studies by J.C.R. Hunt, W.E. Williams (1968) and Y. Kolesnikov, H.Kalis (2021). The magnetohydrodynamic (MHD) flow is characterized by one non-dimensional parameter Hartman (Ha) number. The MHD process is consider in duct cross section:

$$\Omega = \{(x, y, z) : -L \le x \le L, 0 \le y \le C, -\infty \le z \le +\infty\}$$

with electrically non conducting walls. External magnetic field in the (x, y) plane is applied as $\mathbf{B} = \{B_0 cos(\phi_0), B_0 sin(\phi_0), 0\}$,where B_0 is constant value of the magnetic field induction in the direction under angle ϕ_0. An electric current is supplied to duct by two couples of linear electrodes $z \in (-\infty, +\infty)$ at the duct walls $y = 0$ and $y = C$ by $x = \pm a$ perpendicular to the magnetic field. At the walls $y = 0$, $y = C$ in the first electrode $x = -a$ the electric current densities $J_x = 0, J_y = +\infty$, in the second electrode $x = +a$ the electric current densities $J_x = 0, J_y = -\infty$. Therefore $J_y = \delta(x + a) - \delta(x - a)$. We will find the symmetric distribution of azimuthal velocity $U = U(x, y)$ and induced magnetic field $H = H(x, y)(J_x = \frac{\partial H}{\partial y}, J_y = -\frac{\partial H}{\partial x})$ by solving the following boundary value problem for partial differential equations (PDEs):

$$\begin{cases} \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + Ha\left(cos(\phi_0)\frac{\partial H}{\partial x} + sin(\phi_0)\frac{\partial H}{\partial y}\right) = 0, \\ \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + Ha\left(cos(\phi_0)\frac{\partial U}{\partial x} + sin(\phi_0)\frac{\partial U}{\partial y}\right) = 0, \\ x \in (-L, L), y \in (0, C), U(\pm L, y) = H(\pm L, y) = U(x, 0) = U(x, C) = 0, \\ H(x, 0) = H_0(x), H(x, C) = \gamma H_0(x), \end{cases}$$

where the constant $\gamma = -1; 0; 1$ is depending on the direction for the electric current injection in electrodes, $H_0(x)$ is the following function:

$$0, (x \in (-L, -a), x \in (a, L)), -I_0, (x \in (-a, a)),$$

where I_0 characterize the dimensionless value of the electric current ($I_0 = 1$).

Using transformations
$S_{\pm} = H \pm U, U = \frac{S_+ - S_-}{2}, H = \frac{S_+ + S_-}{2}, S_{\pm}(x, y) = W_{\pm}(x, y)exp(-\pm \alpha(f_0(x, y)), f_0(x, y) = xcos(\phi_0) + ysin(\phi_0), \alpha = Ha/2$, we have PDEs problem for functions $W_{\pm}(x, y)$ solved analytically with Fourier method and numerically with Matlab.

FUZZY VOLTERRA INTEGRAL EQUATION OF THE SECOND KIND WITH WEAKLY SINGULAR KERNEL

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Fuzzy numbers are used instead of real numbers when there is some uncertainty in the data. Fuzzy function is a map from a set of real numbers to the set of fuzzy numbers.

We consider a fuzzy Volterra integral equation of the second kind

$$u(t) = f(t) + \int_0^t K(t,s)u(s)ds, \quad t \in [0,T],$$

where $K \colon D_T \to \mathbb{R}$ is a given kernel with domain $D_T = \{(t,s) \colon 0 \le s < t \le T\}$, f is a given fuzzy function and u is an unknown fuzzy function. The case of smooth kernel, $K \in C^2(\overline{D})$, where K may change sign on some lines, is considered in [1]. In the weakly singular case the kernel K is assumed to be in $C^m(D_T)$ for some $m \in \mathbb{N}$ and satisfy

$$\left| \left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^l K(t,s) \right| \le C_{K,m} \begin{cases} 1 & \text{if } j+\alpha < 0, \\ 1 + |\log(t-s)| & \text{if } j+\alpha = 0, \\ (t-s)^{-j-\alpha} & \text{if } j+\alpha > 0, \end{cases}$$

where $j, l \in \mathbb{N}_0$, $j + l \le m$ and $\alpha < 1$.

We consider the case where K is weakly singular and may change sign in D_T on some vertical and/or horizontal lines. Results on the existence, uniqueness, smoothness and fuzziness of the solution are proved.

To solve the integral equation numerically we propose a piecewise spline collocation method with a graded mesh. By increasing the number of collocation points we show that the numerical solution exists and converges to the exact solution. We obtain exact convergence rates depending on smoothness of the solution and on the grading parameter of the mesh and give sufficient conditions for the fuzziness of the approximate solution. We also present some numerical examples.

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INVERSE PROBLEM FOR THE HEAT EQUATION WITH ENERGY CONDITION

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The talk considers the inverse problem for the non-stationary heat equation in a bounded simply connected domain with a specific over-determination condition. The existence of a very weak solution is proved. This is the first step for studying the solvability of the Stokes problem with the prescribed kinetic energy. The motivation for such problems comes from the recent paper of T. Buchmaster and V. Vicol (see [1]).

This is joint work with Konstantinas Pileckas.

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NUMERICAL ANALYSIS OF A FDM SOLUTION TO A FLUID-STRUCTURE INTERACTION PROBLEM

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A fluid-structure interaction model based on [1] was derived together with team members and is briefly presented in this talk. It couples two media with different physical characteristics. For example, it can be used to model blood flow in a small human arteriole or motor oil flow in a long, elastic pipe. Among some important aspects of this model, the inclusion of elasticity and viscosity of both media is stressed out.

Ir particular, a problem modelling blood flow in a small elastic arteriole is the main concern of this talk. By analysing the speed of blood flow averaged over cross-sections of the vessel, we get a 4th order PDE of the following form:

$$c_1 Q_{tt} + c_2 Q_{xxtt} + c_3 Q_t + c_4 Q_{xxt} + c_5 Q_{xxxx} + c_6 Q_{xx} + c_7 Q = 0,$$

here Q is the averaged speed and coefficients c_i include physical characteristics of the model, such as the radius of tube, elastic tube thickness, density of the elastic medium, etc.

The main focus here is on some numerical aspects of the solution to the presented PDE problem. Namely, two schemes of different orders of accuracy are presented to solve it numerically. Their accuracies in time and space are analysed. The stability of proposed numerical schemes is investigated. Results of some interesting numerical tests are discussed to confirm the accuracy of constructed solvers.

The model is implemented for a single pipe geometry and for a Y-shaped network of vessels. Numerical results are compared with the full Navier-Stokes simulations. Numerical experiments confirm that the proposed ansatz is effectively applicable for a broad class of problems.

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STABILITY OF A STEADY CONVECTIVE FLOW IN A VERTICAL CHANNEL WITH PERMEABLE BOUNDARIES

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Consider a tall vertical channel formed by two parallel infinite vertical planes $x = \pm h$. Viscous incompressible chemically reacting fluid is located in the region between the planes. A steady convective flow in the vertical direction is induced by the two factors: (a) nonlinear heat sources (heat is released as a result of a chemical reaction in accordance with the Arrhenius' law [1]), and (b) temperature difference between the walls of the channel. It is assumed that there exists also a steady flow in the direction perpendicular to the planes through the walls of the channel, namely, $U|_{x=\pm h} = \pm U_0$, where U_0 is a constant.

In the present paper linear stability problem for combined steady flow consisting of two velocity components is formulated and solved numerically. Linear stability is investigated using the method of normal modes. The boundary value problem for the system of ordinary differential equations is discretized using Chebyshev collocation method. The nonlinear boundary value problem for the base flow is solved numerically using Matlab. Stability boundary depends on the relationship between four parameters chracterizing the problem: the Prandtl number, the Frank-Kamenetskii parameter, the Reynolds number and the Grashof number. It is shown that for large Prandtl numbers instability is associated with thermal running waves propagating downstream with sufficiently large phase velocity. Calculations show that depending on the values of the parameters there exist regions of stabilization and destabilization in the parameter space. In particular, cross-flow stabilizes the flow while internal heat generation and temperature difference between the walls of the channel has a destabilizing effect.

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IN SEARCH OF DYNAMICAL CHAOS

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Systems of ordinary differential equations of the form

$$X' = F(\mu(WX - \Theta)) - X \tag{1}$$

are considered. Here F is a sigmoidal vector function, W is quadratic matrix with constant entries, μ and Θ are vectors of parameters. First, the three dimensional system is considered with a critical point which has characteristic numbers of the form $\lambda_1 > 0$, $\lambda_{2,3} = \alpha \pm \mathbf{i}\beta$, $\alpha < 0$, $\beta \neq 0$, $\mathbf{i} = \sqrt{-1}$. Such a point, accordingly to [1], is called saddle-focus(2,1). Second, the three dimensional system is considered with a critical point which has characteristic numbers of the form $\lambda_1 < 0$, $\lambda_{2,3} = \alpha \pm \mathbf{i}\beta$, $\alpha > 0$, $\beta \neq 0$. Such a point, accordingly to [1], is called saddle-focus(1,2). These cases are interchangeable by the reversing the independent variable t. The four dimensional system of the form (1) is considered also, where a critical point has the characteristic numbers $\lambda_{1,2} = \alpha_1 \pm \mathbf{i}\beta_1$, $\lambda_{3,4} = \alpha_2 \pm \mathbf{i}\beta_2$, $\alpha_1\alpha_2 < 0$, $\beta_{1,2} \neq 0$. The critical point is called saddle-focus(2,2). The dynamics of such systems essentially depends on the so called *saddle value* σ, which for the four dimensional system is $\sigma = \alpha_1 + \alpha_2$.

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BLOOD VELOCITY COMPUTATION INSIDE OF A HUMAN HEART LEFT ATRIUM APPENDAGE

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The talk is devoted to the numerical study of the Navier-Stokes equations in patient-specific geometry of the human heart left atrium. During atrial fibrillation, the left atrium and left atrium appendage (LAA) play a very important role in thrombogenesis. In the opinion of cardiologists, the reason of thrombogenesis is stagnation of blood flow in some parts of LAA, which likely depends on its geometry. Numerical solutions of the Navier-Stokes equations give a real patient blood flow velocity inside of LAA and detect the stagnation zones. The obtained results could help to prepare for surgical treatment and to make a conclusion about whether anticoagulants must be prescribed.

The results are obtained in collaboration with Sigita Aidietienė, Audrius Aidietis, Oleg Ardatov, Sergejus Borodinas, Rimgaudas Katkus, Kristina Kaulakytė, Grigory Panasenko, Konstantinas Pileckas.

APPROXIMATION BY THE MELLIN TRANSFORM OF THE RIEMANN ZETA-FUNCTION

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Let $\zeta(s)$, $s = \sigma + it$, denote the Riemann zeta-function. By the Voronin theorem the function $\zeta(s)$ is universal in the sense that its shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$, approximate a wide class of analytic functions.

In the theory of $\zeta(s)$, in particular, for investigations of the moments, the Mellin transforms

$$\mathcal{Z}_k(s) = \int_1^\infty \left| \zeta\left(\frac{1}{2} + ix \right) \right|^{2k} x^{-s}\, \mathrm{d}x, \quad k \in \mathbb{N},$$

are widely used. It turns out that the functions $\mathcal{Z}_k(s)$ also have good approximation properties. In the report, we consider the function $\mathcal{Z}(s) \overset{\text{def}}{=} \mathcal{Z}_1(s)$.

Let $\delta > 0$ be a small fixed number, and $\Delta = \{s \in \mathbb{C} : 1/2 + \delta < \sigma < 1 - \delta\}$. Denote by $H(\Delta)$ the space of analytic on Δ functions endowed with the topology of uniform convergence on compacta. Our main result is the following statement.

THEOREM 1. *There exists a closed non-empty set $F \subset H(\Delta)$ such that, for every compact set $K \subset \Delta$, $f(s) \in F$, and every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\mathcal{Z}(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

Here $\mathrm{meas}A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Unfortunately, the set F can't be given explicitly.

Theorem 1 is a consequence of a limit theorem in the space $H(\Delta)$ for the function $\mathcal{Z}(s)$. Let $\mathcal{B}(H(\Delta))$ denote the Borel σ-field of the space $H(\Delta)$.

THEOREM 2. *On $(H(\Delta), \mathcal{B}(H(\Delta)))$, there exists a probability measure P, such that*

$$\frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \mathcal{Z}(s + i\tau) \in A \right\}, \quad A \in \mathcal{B}(H(\Delta)),$$

converges weakly to P as $T \to \infty$.

The functions $\mathcal{Z}_k(s)$ were introduced in [1].

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STUDY OF TWO COMBINED HEAT, MASS TRANSFER AND REACTION MODELS WITH APPLICATIONS TO BIOMASS PELLET COMBUSTION

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Extensive research has been devoted to simulating gasification or combustion processes [1],[2] in isotropic heterogeneous materials such as wood, wood briquettes, or pellets. Biomass pellets composed of wood, straw, peat, etc are a promising alternative to fossil fuels in many applications such as home heating. In order to further increase the efficiency while reducing harmful emissions, smart control methods have been proposed, e.g. using external electric and magnetic fields. If heat and mass transfer processes within the pellets are to be taken into account and the controlling device is to be of low computational power, then minimalistic mathematical models are required.

A network model has been previously proposed by the authors [3] however, the results have not been validated against more complete models. We develop an alternative model preserving the topology of the network model, consisting of nodes and channels connecting the nodes, with one-dimensional gas dynamics equations governing the gas flow between the nodes. The main object of research in the developed model is the development of a thermal gasification dynamics model, where the primary task is to simplify a real 3-dimensional model to a 1-dimensional model by discretizing pressure and flow with the help of a graph model validation and simulation. Mass conservation laws are used at the nodes to couple the gas dynamics equations on the various nodes. The resistance to gas flow between the nodes in the two models is described by different parameters permeability coefficient for the simple network model and channel length and diameter for the alternative model; these can be adjusted to apply both methods to a certain problem. The results of the two models are compared for some representative geometries.

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A METHOD BASED ON CENTRAL PART INTERPOLATION FOR FRACTIONAL DIFFERENTIAL EQUATIONS

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We consider a fractional initial value problem

$$(\mathrm{D}_{Cap}^{\alpha_2}u)(t)+r_1(t)(\mathrm{D}_{Cap}^{\alpha_1}u)(t) + r_0(t)u(t) = f(t), \quad t \in [0,1], \tag{1}$$

$$u(0) = u_0, \quad u_0 \in \mathbb{R} := (-\infty, \infty), \tag{2}$$

where $0 < \alpha_1 < \alpha_2 < 1$, and the functions r_0, r_1 and f are continuous: $r_0, r_1, f \in C[0,1]$. Here $\mathrm{D}_{Cap}^{\alpha}u$ is an α-order Caputo fractional derivative of the unknown function $u = u(t)$. Then (see [1]) problem (1)–(2) possesses a unique solution $u \in C[0,1]$ such that $\mathrm{D}_{Cap}^{\alpha_2}u \in C[0,1]$. However, we cannot in general expect the solution u to belong to $C^q[0,1]$ (that is, u is q times continuously differentiable on interval $[0,1]$) for r_0, r_1 and $f \in C^q[0,1]$, $q \in \mathbb{N} := \{1, 2, \ldots\}$. Instead, we can show that if $r_0, r_1, f \in C^{q,\mu}(0,1]$, $q \in \mathbb{N}$, $\mu \in \mathbb{R}$, $\mu < 1$, then u and its derivative $\mathrm{D}_{Cap}^{\alpha_2}u$ belong to $C^{q,\nu}(0,1]$, where $\nu = \max\{1-(\alpha_2-\alpha_1),\mu\}$ [1]. Here, by $C^{q,\mu}(0,1]$ ($q \in \mathbb{N}$, $\mu \in \mathbb{R}$, $\mu < 1$) we denote the set of functions $u \in C[0,1] \cap C^q(0,1]$ such that

$$|u^{(i)}(t)| \le c \left\{ \begin{array}{ll} 1 & \text{if } i < 1 - \mu \\ 1 + |\log t| & \text{if } i = 1 - \mu \\ t^{1-\mu-i} & \text{if } i > 1 - \mu \end{array} \right\}, \quad 0 < t \le 1, \quad i = 1, \ldots, m,$$

where c is a positive constant independent of t. Thus, when constructing high order numerical methods for problem (1)-(2), one should take into account, in some way, the possible non-smooth behaviour of the exact solution.

We propose a method for solving (1)-(2) based on improving the boundary behavior of the exact solution with the help of a change of variables, and on central part interpolation by polynomial splines on the uniform grid, which was introduced in [2] for the numerical solution of weakly singular Fredholm integral equations of the second kind. We adapt this approach for the problem (1)-(2) and derive global error estimates for the approximate solution [3].

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A GENERALIZED BOHR-JESSEN TYPE THEOREM FOR THE EPSTEIN ZETA-FUNCTION

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In the talk, some aspects on the value-distribution of the Epstein zeta-function will be discussed. Let Q be a positive definite quadratic $n \times n$ matrix and $Q[\underline{x}] = \underline{x}^{\mathrm{T}} Q \underline{x}$ for $\underline{x} \in \mathbb{Z}^n$. The Epstein zeta-function $\zeta(s; Q)$, $s = \sigma + it \in \mathbb{C}$, is defined, for $\sigma > \frac{n}{2}$, by the series

$$\zeta(s; Q) = \sum_{\underline{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}} (Q[\underline{x}])^{-s},$$

and can be continued analytically to the whole complex plane, except for a simple pole at the point $s = \frac{n}{2}$ with residue $\frac{\pi^{n/2}}{\Gamma(n/2)\sqrt{\det Q}}$. The probabilistic limit theorem of Bohr-Jessen type for the Epstein zeta-function $\zeta(s; Q)$ with even $n \geq 4$ and integers $Q[\underline{x}]$ was published in [1]. Namely, we have proved that, on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists an explicitly given probability measure $P_{Q,\sigma}$ such that, for $\sigma > \frac{n-1}{2}$,

$$\frac{1}{T} \mathrm{meas}\{t \in [0, T] : \zeta(\sigma + it; Q) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}), \tag{1}$$

converges weakly to $P_{Q,\sigma}$ as $T \to \infty$. Here $\mathrm{meas} A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and $\mathcal{B}(\mathbb{C})$ – the Borel σ-field of the space $\mathbb{C}$.

We will discuss more general case, i. e., in place of (1), the weak convergence for

$$\frac{1}{T} \mathrm{meas}\{t \in [0, T] : \zeta(\sigma + i\varphi(t); Q) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

where $\varphi(t)$ is a differentiable function with a monotonic derivative, will be analyzed. This result was obtained jointly with A. Laurinčikas in [2].

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SUPER-RESOLUTION DIFFUSION WEIGHTED IMAGING FROM HIGHLY COMPRESSIVELY SENSED MEASUREMENTS

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The purpose of this paper is to present a new, rapid compression-sensed magnetic resonance image reconstruction and enhancement technique for diffusion imaging. This study's primary objective is to eliminate two significant obstacles of diffusion MR image reconstruction techniques, i.e. combining a highly-sparse k-q space sampling pattern with super-resolution (SR) image enhancement circumvents the limitations of image resolution and algorithm reconstruction time. The algorithm utilizes the Wasserstein Generative Adversarial Networks (WGAN) framework. Moreover, the proposed method combines a highly compressively sensed k-space with deformable motion registration module to fuse adjacent frames in order to effectively use detailed information in multiple consecutive frames, and improves the spatio-temporalality of low-resolution images in sequential images. WGANs are initially taught to map zero-filling images onto full-sample images. Then, the zero-filled portion of the k-space data is retrieved [1]. The proposed method was introduced to optimize model training while GAN was used to improve the effect of image high-frequency texture detail reconstruction [2]. The $\Psi_G = \{W_{1:L}; b_{1:L}\}$ represents the weight and deviation of the L-layer-deep network and is obtained by optimizing the SR generation network's loss function l_G [3].

$$\Psi_G^* = \underset{\Psi_G}{argmin} \frac{1}{N} \sum_{t=1}^{N} l_G(G_{\Psi_G} I_t^{LR}, I_t^{HR}).$$

The loss function of the entire network, according to the SRGAN, includes the loss functions of the generator l_G and discriminator l_D:

$$l_D = \frac{1}{N} \sum_{n=1}^{N} \Big(log\big(1 - D_{\Psi_D}(G_{\Psi_G} I^{SR})\big)\Big) - log\Big(D_{\Psi_D}(I^{HR})\Big).$$

Here, N is the number of target images, and $D_{\Psi_D}(G_{\Psi_G} I^{SR}))$ and $D_{\Psi_D}(I^{HR})$ are the reconstructions for the discriminator's generator $G_{\Psi_G} I^{SR})$ and the original image I^{HR}, respectively.

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MODELING THE EUROPEAN UNION EMISSION TRADING PRICES TIME-SERIES BY GENERATIVE ADVERSARIAL NETWORKS WITH INBUILT COMPRESSED SENSING PARADIGM BASED TRAINING SETS DENOISING

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Carbon prices follow a stochastic process of complex time series with nonstationary and nonlinear properties, making the forecasting of $CO2$ emission prices a crucial and difficult issue for policy-makers and market participants.The presented method combines the generative adversarial networks with the compressed sensing based test and training sets denoising paradigm. Autoencoders use unsupervised learning to extract features from high-dimensional data and reconstruct them from representations of those features. They consist of encoding and decoding procedures, with encoding mapping the input to the feature spaces and decoding returning the spaces to their original state. There are two NNs for the GAN structure: Generator (G) and Discriminator (D). The G and D engage in a competition wherein G develops candidates and D evaluates them. The purpose of G's training is to enhance D's error rate. The training could be described by the classical value function in the way below:

$$\min_G \max_D = E_{x \ p_{data}(x)} \left[\log \left(D\left(x\right) \right) \right] + E_{z \ p_z(z)} \left[\log \left(1 - D\left(G\left(z\right) \right) \right) \right].$$

The G is a differentiable function represented by a multilayer perceptron (MLP), and $D(x)$ is the probability that x originates from the data rather than p. Simultaneously, the maximization of assigning correct labels for D and the minimization of $log \left(1 - D\left(G\left(z\right) \right) \right)$ are trained.

The algorithm also utilizes the Compressed Sensing theorem [3]. Initial data filtering takes advantage of sparsity. This is based on the assumption that orthogonal data is redundant. The CSD is first carried out using the ANN-learning paradigm [1]. It has been shown that the GAN based forecasting scheme combined with modified learning paradigm is an excellent method for enhancing model prediction performance by reducing the level of noise detected in EU ETS price data.

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FUZZY LOGIC IN SOME MACHINE LEARNING METHODS

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In this work, the usage of fuzzy equivalence relations aggregation to determine the similarities of objects in the clustering process will be shown [1]. Different t-norms, such as Lukasiewicz t-norm, product t-norm, and Hamacher t-norm, are used. It is possible to assign different weights to attributes, thus defining the importance of attributes in the decision-making process. Also, we will demonstrate how membership functions can be used in classification problems to calculate the degree of membership of each data point in each class. This allows for making classification decisions based on these membership values. Such an approach enables a more nuanced and flexible classification process that can take into account the inherent uncertainty and variability present in many real-world datasets. Membership functions can be employed in various classification algorithms, such as fuzzy k-nearest neighbors, fuzzy decision trees, and fuzzy neural networks. These methods can provide improved performance in situations where traditional classification techniques may struggle due to data uncertainty or imprecision. Furthermore, the work presents the performance of these methods on different datasets.

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JOINT APPROXIMATION BY LERCH ZETA-FUNCTIONS

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Let $s = \sigma + it$ be a complex variable, and $0 < \alpha \leqslant 1$ and λ be real parameters. The Lerch zeta-function $L(\lambda, \alpha, s)$ is defined, for $\sigma > 1$, by the series $L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} e^{2\pi i \lambda m}(m + \alpha)^{-s}$, and has the meromorphic continuation to the whole complex plane. The function $L(\lambda, \alpha, s)$, as other zeta-functions, for some classes of the parameters λ and α, is universal in the sense that its shifts $L(\lambda, \alpha, s + i\tau)$, $\tau \in \mathbb{R}$, approximate every analytic function defined on the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Some results on approximation of analytic functions by shifts $L(\lambda, \alpha, s + i\tau)$ for all parameters λ and α were obtained in [1] and [2].

In the report, we generalize the above results for a collection of Lerch zeta-functions $L(\lambda_1, \alpha_1, s)$, $\ldots, L(\lambda_r, \alpha_r, s)$ with arbitrary parameters $\lambda_1, \ldots, \lambda_r$ and $\alpha_1, \ldots, \alpha_r$. Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_r)$, $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$, and $H(D)$ be the space of analytic on D functions equipped with the topology of uniform convergence on compacta.

THEOREM 1. *Suppose that $0 < \lambda_j \leqslant 1$ and $0 < \alpha_j \leqslant 1$ are arbitrary, $j = 1, \ldots, r$. Then there exists a non-empty closed set $F_{\underline{\lambda}, \underline{\alpha}} \subset H^r(D)$ such that, for compact sets $K_1, \ldots, K_r \subset D$, $(f_1(s), \ldots, f_r(s)) \in F_{\underline{\lambda}, \underline{\alpha}}$, and every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |L(\lambda_j, \alpha_j, s + i\tau) - f_j(s)| < \varepsilon \right\} > 0.$$

Theorem 1 has a discrete version. Let $\underline{h} = (h_1, \ldots, h_r)$, $h_j > 0$, $j = 1, \ldots, r$.

THEOREM 2. *Under hypotheses of Theorem 1, there exists a non-empty closed set $F_{\underline{\lambda}, \underline{\alpha}, \underline{h}} \subset H^r(D)$ such that, for compact sets $K_1, \ldots, K_r \subset D$, $(f_1(s), \ldots, f_r(s)) \in F_{\underline{\lambda}, \underline{\alpha}, \underline{h}}$, and every $\varepsilon > 0$,*

$$\liminf_{N \to \infty} \frac{1}{N + 1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |L(\lambda_j, \alpha_j, s + i k h_j) - f_j(s)| < \varepsilon \right\} > 0.$$

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VISCOELASTIC MODELS OF TISSUES; MATHEMATICAL BACKGROUND AND HYSTERESIS LOOPS

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Human, animal and plant tissues are typically viscoelastic. Their response to a mechanical load is neither purely viscous, nor purely elastic, but somewhere between. We say they exhibit both viscous and elastic properties. The models standing behind such materials are called viscoelastic models. Their mechanical behaviour is governed by stress σ and strain ε relation, called physical or constitutive equation, as well. Physical relations of (H) - Hookean elastic and (N) Newtonian viscous members are respectively: (H): $\sigma = E\varepsilon$ and (N): $\sigma = \eta\frac{d\varepsilon}{dt}$. In viscoelastic models, stress and strain are both functions of time. Parallel "|" and/or serial "-" connection is exploited when creating various viscoelastic models. Arisen configurations then induce corresponding geometric equations. Geometric equations are of a great importance herein: In *parallel connection* of (H) and (N), both members move equally, $\varepsilon = \varepsilon_H = \varepsilon_N$; summing up of stress functions of involved members yields the total stress function of the model $\sigma = \sigma_H + \sigma_N$. On the other hand, from *serial connection* we have geometric equations $\varepsilon = \varepsilon_H + \varepsilon_N$ and $\sigma = \sigma_H = \sigma_N$. Stress-strain relation of entire model can be then derived from corresponding geometric and physical equations by eliminating all variables indexed by letters H and N. In viscoelastic case, it is always a differential equation. Moreover, the difference between the mechanical response during loading and unloading is called hysteresis, [1]. Tissues can be represented by less or more complex viscoelastic models. As an example, human lung parenchyma can be taken – with the model standing behind it, of its structural form, [2]

$$(H_2)|[(H_1) - (N_1)]$$

and constitutive relation

$$\eta_1(E_1 + E_2)\frac{d\varepsilon(t)}{dt} + E_1 E_2 \varepsilon(t) = \eta_1 \frac{d\sigma(t)}{dt} + E_2\sigma(t).$$

Having the viscoelastic model at hand, the theoretical investigation can be done, validated with lab tests and used for behaviour prediction of materials' response to various load.

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COMPARATIVE ANALYSIS OF MODELS OF GENETIC AND NEURONAL NETWORKS

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Genetic networks can me modeled by the system of ordinary differential equations

$$\begin{cases} x_1' = f_1(w_{11}x_1, ..., w_{1n}x_n, \theta_1) - v_1 x_1, \\ ..., \\ x_n' = f_n(w_{n1}x_1, ..., w_{nn}x_n, \theta_n) - v_n x_n, \end{cases}$$

where $f_i(w_{i1}x_1, ..., w_{in}x_n) = e^{-e^{-\mu_i(w_{i1}x_1 + ... + w_{in}x_n - \theta_i)}}$ are Gompertz functions.

In the theory of neuronal networks the system of ODE

$$\begin{cases} x_1' = \tanh(a_{11}x_1 + ... + a_{1n}x_n) - b_1 x_1, \\ ..., \\ x_n' = \tanh(a_{n1}x_1 + ... + a_{nn}x_n) - b_n x_n, \end{cases}$$

is used to model a network.

We make comparative analysis of both systems for particular cases of dimensions 2, 3 and 4. In a focus of the research are attractors of both systems. We construct examples of attractors of different nature, including stable equilibria and limit cycles.

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AGGREGATION OF CREDIT RISK LEVELS BASED ON FUZZY CLUSTERING

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Our research deals with a special construction of aggregation, which is based on fuzzy clustering. The need for aggregation operators based on similarities appears, for example, in decision making if values to be aggregated represent evaluations provided by several experts. In this talk we develop and apply in risk management the approach proposed in our previous works [1], [2] where such similarities were described by fuzzy equivalence relations.

Suppose we have an initial data set (learning set) of customers with different attributes that somehow characterize these customers. Suppose the experts have assessed the levels of credit risk for these customers using their own analytical approaches and skills. We propose a special design aimed at involving the results of fuzzy clustering in the aggregation of risk levels given by the experts. For this aim, fuzzy clustering of the learning set based on customer attributes is used. The inclusion of risk level assessments given by the experts for all customers of a cluster in the process of aggregation of risk level assessments given by the experts for one object of this cluster allows to obtain a more objective and unbiased final aggregated credit rating.

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VALIDATION OF THE TWO-SAMPLE LOCATION-SCALE MODEL

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The two sample problems probably are the most common in statistics. In practice for comparison of two groups usually some measure of central tendency is compared, for example, using t-test or Wilcoxon test. However, the comparison of the whole distribution would be preferable and could lead to more subtle conclusions on the observed differences between two groups compared to the above mentioned two-sample tests.

In medicine to declare a positive treatment effect for all population members we could test for the location model $F_1(t) = F_2(t - \mu)$, $t \in \mathbb{R}$, between two distribution functions F_1 and F_2. It indicates a uniform shift between two samples by some constant μ. Or else, if the equality of two means or medians is not rejected, there could still be the differences between the distributions of both samples regarding their scales, indicating non-equal treatment effect for all members of the population. The scale model between two distribution functions is expressed as $F_1(t) = F_2(t/\sigma)$, $t \in \mathbb{R}$, where σ is a scale parameter. Taken both mentioned semi-parametric models together, the two-sample location-scale model is defined as

$$F_1(t) = F_2\left(\frac{t - \mu}{\sigma}\right), \quad t \in \mathbb{R}.$$

Numerous non-parametric tests for simultaneous test of the location and scale differences under the the two-sample location-scale model can be found in the literature (see for example [1]). In order to apply those tests the two-sample location-scale model needs to be validated beforehand. Doksum and Sievers [2] introduced simultaneous confidence bands for the general shift function, which is a graphical tool for validation of the location and the location-scale models. Hall et.al. [3] proposed a test based on the empirical characteristic functions to verify if the two samples belong to the same location-scale model family.

We propose to use the empirical likelihood-based test for the probability-probability plot to validate the location-scale model between two samples. This can be done by testing the equality of both distributions using the location and scale parameter estimation to transform one of the samples under the location-scale assumption.

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ASYMPTOTIC ANALYSIS AND NUMERICAL SOLUTIONS OF STATIONARY FLOWS IN NETWORK OF VESSELS

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The talk briefly presents the results of asymptotic analysis for non-Newtonian flows in thin tube structures [1], [2]. The computation of the leading term approximation of the solution is related to the equation on the graph, which is an elliptic nonlinear problem. We introduce a numerical method to solve the equation on the graph and apply it to the realistic network of vessels. The results are obtained in collaboration with Kristina Kaulakyte, Nikolajus Kozulinas, Konstantinas Pileckas, Vytenis Sumskas.

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NUMERICAL SOLUTION OF LINEAR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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We consider a class of fractional weakly singular integro-differential equations

$$
\begin{aligned}
(D_{Cap}^{\alpha_p}y)(t) + \sum_{i=0}^{p-1} d_i(t)(D_{Cap}^{\alpha_i}y)(t) \\
+ \sum_{i=0}^{q-1} \int_0^t (t-s)^{-\kappa_i} K_i(t,s)(D_{Cap}^{\theta_i}y)(s)ds = f(t), \quad 0 \le t \le b,
\end{aligned} \tag{1}
$$

subject to the initial conditions

$$
y^{(i)}(0) = \gamma_i, \quad i = 0, \ldots, n-1. \tag{2}
$$

Here D_{Cap}^{δ} is the Caputo differential operator of order $\delta > 0$ and $n := \lceil \alpha_p \rceil$ is the smallest integer greater or equal to the highest fractional order α_p. We assume that $0 \le \alpha_0 < \alpha_1 < \cdots < \alpha_p \le n$, $0 \le \theta_j < \alpha_p$, $0 \le \kappa_j < 1$ ($j = 0, \ldots, q-1$) with $p, q \in \{1, 2, \ldots\}$, and that the given functions d_i ($i = 0, \ldots, p-1$), K_j ($j = 0, \ldots, q-1$) and f are continuous on their respective domains.

Following [1], we reformulate (1)–(2) as a Volterra integral equation of the second kind with respect to the fractional derivative $D_{Cap}^{\alpha_p}y$. We then regularize the solution by a suitable smoothing transformation and solve the transformed integral equation by a piecewise polynomial collocation method on a mildly graded or uniform grid. We show the convergence of the proposed algorithm and present global superconvergence results for a class of specific collocation parameters. Finally, we complement the theoretical results with some numerical examples.

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HOMOGENIZATION OF ALTERNATING NONLINEAR BOUNDARY CONDITIONS, WITH HIGH CONTRASTS

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We address a homogenization problem for the Laplace operator posed in a bounded domain of the upper halfspace, a part of its boundary being in contact with the plane. On this part, the boundary conditions alternate from Neumann to nonlinear-Robin, being of Dirichlet type outside. The nonlinear-Robin boundary conditions are imposed on small regions periodically placed along the plane and contain a Robin parameter that can be very large. Depending on the different relations between parameters (period, size of the regions and Robin parameter), a nonlinear capacity term may arise in the strange term which depends on the macroscopic variable and allows us to extend the usual capacity definition to semilinear boundary conditions, cf. [2]. Extensions to strainer Winkler foundations are considered, focusing mainly on the averaged Robin-Winkler boundary condition, cf. [1] and [3].

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METHOD OF F-TRANSFORMS TO THE VOLTERRA INTEGRAL EQUATION WITH A WEAKLY SINGULAR KERNEL

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The linear Volterra equation whose general form is

$$u(t) = g(t) + \lambda \int_0^t k(t,s)u(s)ds, \qquad t \in [0,1], \tag{1}$$

belongs to the group of classical fundamental equations. In (1), $\lambda \in \mathbb{R} \setminus \{0\}$, *kernel* $k : [0,1] \times [0,1] \to \mathbb{R}$, and *source* $g : [0,1] \to \mathbb{R}$ are given, and $u : [0,1] \to \mathbb{R}$ is unknown. If k is integrable and bounded and $g \in L^2[0,1]$, then (1) always has a unique solution.

We will focus on *weak singular kernels*, which are represented by $K(t,s) = \alpha(t,s)|t-s|^{-\nu}$, where $0 < \nu < 1$, and $\alpha : [0,1] \times [0,1] \to \mathbb{R}$ is smooth above the diagonal. Generalizing, we say that a kernel K is weakly singular [1], if it is absolutely integrable with respect to s and satisfies $\sup_{[0,1]} \int_0^1 |k(t,s)| ds < \infty$.

In the case of (1) with a weakly singular kernel the main problem in the development of numerical methods is the discontinuity of the kernel on the diagonal of the integration domain. In this contribution, we show that in this particular case the (fuzzy) F-transform method [2] is applicable and occupies a special place among other combined methods.

In more detail, to find a numerical solution of (1), all involved functional objects are replaced by their approximations in the form of inverse F-transforms. After this step, integration in (1) becomes applicable only to the basic functions of a uniform fuzzy partition, which does not depend on any particular type of integrand. As a result, the action of integration in (1) is focused on what can be associated with a certain spatial structure (fuzzy partition), and separated from the functional objects in (1).

We have calculated the operational matrix of the Volterra operator by applying the integral operator to the basic functions of a fuzzy partition, and reduced (1) to a system of linear equations with a (non-degenerate) triangular matrix of coefficients. An easily found solution to this system gives the F-transform components of the unknown function and its approximation in the form of the inverse F-transform.

We proved the convergence theorem and gave estimations of the numerical complexity of the proposed method.

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ON THE CONVERGENCE OF THE DIFFERENCE SCHEME FOR NONLINEAR ELLIPTIC EQUATION WITH INTEGRAL BOUNDARY CONDITION

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We investigate the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y, u), \quad (x, y) \in \Omega = \{0 < x < 1, 0 < y < 1\},$$

$$u(x, 0) = \mu_1(x), \quad u(x, 1) = \mu_2(x), \quad u(0, y) = \mu_3(y),$$

$$u(1, y) = \gamma \int\limits_{\xi}^{1} u(x, y)dx + \mu_4(y).$$

It was analyzed with which values of parameters ξ and γ the matrix of the system of difference equations is an M–matrix. Applying the M–matrix theory, the convergence conditions of iterative methods for solving the system of nonlinear difference equations were found. The main result of the investigation is that the majorant function is constructed and the convergence of the finite difference method is proved.

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MODELING OF PHOTONIC CRYSTAL SURFACE-EMITTING LASERS

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Semiconductor diode lasers are small, efficient, and relatively cheap devices used in many modern applications. Multiple applications require emission powers exceeding several ten Watts from a single diode and up to a few kiloWatts from a combined laser system. In this presentation, we consider novel Photonic Crystal Surface-Emitting Lasers (PCSELs), which, in contrast to conventional high-power edge-emitting broad-area lasers (BALs), are capable of emitting high power (up to 80 W at the moment [1]) beams of nearly perfect quality in the (z) direction, perpendicular to the (x/y) plain of active material. The critical part of PCSELs, enabling an efficient coupling of within the active layer generated optical fields and their redirection along the z axis, is a properly constructed 2-dimensional photonic crystal layer. The model to be considered and integrated numerically [2]) is derived from Maxwell equations and is a 1 (time)+2 (space) dimensional system of PDEs for complex optical fields $u(t,x,y) = \binom{u^+}{u^-}$, $v(t,x,y) = \binom{v^+}{v^-}$, and real carrier density $N(t,x,y)$:

$$\frac{1}{v_g}\frac{\partial}{\partial t}\binom{u}{v} = \mathcal{H}(\beta(N))\binom{u}{v} + F_{sp}, \quad \frac{\partial}{\partial t}N = D\nabla^2_{x,y}N + \mathcal{N}(N,|u|^2,|v|^2), \quad (x,y)\in[0,L]\times[0,L],$$

$$\text{where} \quad \mathcal{H}(\beta(N)) = -\begin{pmatrix} \sigma\frac{\partial}{\partial x} & \mathbf{0} \\ \mathbf{0} & \sigma\frac{\partial}{\partial y} \end{pmatrix} + \mathcal{I}\beta(N) + \mathbf{C}, \quad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{cases} u^+|_{x=0} = u^-|_{x=L} = 0, \\ v^+|_{y=0} = v^-|_{y=L} = 0. \end{cases}$$

Besides, an efficient location of several leading modes (Λ,Θ) of the related spectral problem [3],

$$\mathcal{H}(\bar{\beta})\Theta(x,y) = \Lambda\Theta(x,y), \quad \Theta = \begin{pmatrix} \Theta_u \\ \Theta_v \end{pmatrix}, \quad (\Theta_u,\Theta_v) \quad \text{satisfies boundary conditions,}$$

given by a system of four 2-D PDEs, is of great importance when designing PCSEL devices. We shall address the biggest challenges arising when treating above stated problems. Namely, a nontrivial construction of the implicitly defined 4×4 field coupling matrix $\mathbf{C}$, requiring a solution of the Helmholtz problem and multiple integrations of the calculated mode profile with different separately constructed exponentially growing and decaying Green's functions, as well as dealing with large discrete problems relating up to several million variables in large-emission-area (large L) PCSELs.

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THE WADA INDEX, COEXISTING ATTRACTORS, AND SECRET IMAGES

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The lakes of Wada are three (or more) disjoint connected open sets of the plane that all have the same boundary [1]. The visualization of Wada basins for several coexisting attractors in a form of a colormap where every point in the phase space of initial conditions is assigned to a different color became an important topic in computational analysis of nonlinear systems [2]. The uncertainty of a response to a perturbation of a nonlinear system possessing the Wada property (which is higher if compared to a fractal basin boundary) can be quantified by means of the Wada index [3]:

$$\omega_q^{(s)}(p_1, ..., p_m) = \frac{m}{\log(q)} I_q^{(s)} e^{(s)},$$

where s is the size of the observation window; q is the current color; m is the number of different colors in the current observation window; p_k is the discrete probability of the k-th color in the observation window, $k = 1, 2, ..., m$; $I_q^{(s)}$ is the indicator function of color q in the observation window; $e^{(s)}$ is the Shannon entropy of the current observation window.

The introduction of the concept of the perfect covering of the stego image helps to design the image sharing scheme based on the Wada index (the Wada index is directly exploited in the decoding scheme) [4]. A covering is a perfect covering if a sequential modification at the current observation window does not cause conflicts in all previous observation windows. It can be shown that a perfect covering does not exist for a carrier image with periodic boundary conditions. Also, it appears that the encoded stego image does not depend on the type of a perfect covering [4].

The proposed image hiding scheme based on the Wada index ensures that the secret image is not leaked from the modified bit planes with lowest indexes, the stego image is robust against RS steganalysis algorithms, and the payload capacity of the carrier image is comparable to best LSB schemes [4].

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HEURISTIC CHOICE OF THE REGULARIZATION PARAMETER IN TIKHONOV METHOD

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We consider an operator equation

$$Au = f_*, \quad f_* \in R(A),$$

where $A \in L(H, F)$ is the linear continuous operator between real Hilbert spaces H and F. We assume that instead of the exact right-hand side f_* we have only an approximation $f \in F$. To get the regularized solution we consider Tikhonov method $u_\alpha = (\alpha I + A^*A)^{-1}A^*f$, where $\alpha > 0$ is the regularization parameter.

Well-known heuristic rule for choosing regularization parameter is the quasioptimality principle, where the parameter is chosen as the global minimum point of the function

$$\psi_Q(\alpha) = \alpha \left\| du_\alpha/d\alpha \right\| = \alpha^{-1} \left\| A^*(Au_{2,\alpha} - f) \right\|, \quad u_{2,\alpha} = (\alpha I + A^*A)^{-1}(\alpha u_\alpha + A^*f),$$

on the set of parameters $\Omega = \left\{ \alpha_j : \alpha_j = q\alpha_{j-1}, j = 1, 2, ..., M, 0 < q < 1 \right\}$. Unfortunately, this rule is unstable in this sense that it often fails in case of heat-type problems.

To get stable parameter choice rule we introduce modified quasioptimality criterion function $\psi_{\mathrm{MQ}}(\alpha)$ in the form:

$$\psi_{\mathrm{MQ}}(\alpha_0) = \psi_{\mathrm{Q}}(\alpha_0),$$

$$\psi_{\mathrm{MQ}}(\alpha_j) = \max\{\psi_{\mathrm{Q}}(\alpha_j), (d_{MD}(\alpha_j)/d_{MD}(\alpha_{j-1}))^{2/3}\psi_{\mathrm{MQ}}(\alpha_{j-1})\}, j > 0,$$

where the function $d_{MD}(\alpha) = \left\| B_\alpha(Au_\alpha - f) \right\|$, $B_\alpha = \alpha^{1/2}(\alpha I + AA^*)^{-1/2}$. Choosing global minimum point of the function $\psi_{\mathrm{MQ}}(\alpha)$ as regularization parameter gives us stable heuristic rule, but to get more accuracy we present the following algorithm.

In article [1] is shown that at least one local minimum point m_k of the quasioptimality criterion function is always a good regularization parameter. Let α_{MQ} be the global minimum point of the function $\psi_{\mathrm{MQ}}(\alpha)$ on the set of local minimum points of the function $\psi_Q(\alpha)$.

Heuristic rule. For the regularization parameter choose the parameter

$$\alpha_H = \max\{\alpha_Q, \min\{\alpha_{MQ}, \alpha_{HR}\}\},$$

where α_Q and α_{HR} are the global minimum points of the functions $\psi_Q(\alpha)$ and $\psi_{HR}(\alpha) = \alpha^{-1/2}d_{MD}(\alpha)$, respectively.

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DYNAMIC EQUATIONS ON TIME SCALES

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In 1988 (Ph.D. thesis), Stefan Hilger [1] introduced the calculus of time scale in order to unify continuous and discrete analysis, see [2].

The equivalence (conjugacy) and the linearization problem in the theory of differential and difference equations (homeomorphisms, noninvertible mappings) was explored by many mathematicians.

In the present talk we generalize the reduction principle to the case of the dynamic systems on time scale in Banach space whose linear part split into two parts and satisfy weaker condition than exponential dichotomy.

We consider the dynamic system in a Banach space on unbounded above and unbounded below time scales

$$\begin{cases} x^\Delta &=& A(t)x + f(t, x, y), \\ y^\Delta &=& B(t)y + g(t, x, y). \end{cases} \tag{1}$$

This system satisfies the conditions of integral separation with the separation constant ν, nonlinear terms are unbounded and ε-Lipshitz, and the system has a trivial solution. Based on the integral version of the idea developed in the article [3] we generalize and find sufficient condition under which the system (1) is dynamic equivalent to

$$\begin{cases} x^\Delta &=& A(t)x + f(t, x, u(t, x)), \\ y^\Delta &=& B(t)y + g(t, v(t, y), y). \end{cases} \tag{2}$$

In the case of unbounded above time scales we also find sufficient condition under which the noninvertible system is dynamic equivalent to

$$\begin{cases} x^\Delta &=& A(t)x + f(t, x, u(t, x)), \\ y^\Delta &=& B(t)y + g(t, k(t, x, y), y). \end{cases} \tag{3}$$

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VOLTERRA INTEGRODIFFERENTIAL EQUATIONS ON TIME SCALES

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Consider Volterra integrodifferential equation on an arbitrary unbounded time scale $\mathbb{T}$

$$x^{\Delta}(t) = f(t) + \int_a^t K(t, s, x(s), x^{\Delta}(s))\Delta s, \quad x(a) = x_0.$$

We define a new Volterra integral equation

$$z(t) = F(t) + \int_a^t k_1(t, s, z(s))\,\Delta s, \quad a, t \in I_{\mathbb{T}} = [a, +\infty) \cap \mathbb{T}, \tag{1}$$

where $z \colon I_{\mathbb{T}} \to \mathbb{R}^{2n}$ is the unknown function, and $k_1 \colon I_{\mathbb{T}} \times I_{\mathbb{T}} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be rd-continuous in its first and second variable. Equation (1) is known as a Volterra integral equation on time scales. Let $F \colon I_{\mathbb{T}} \to \mathbb{R}^{2n}$, $L \colon I_{\mathbb{T}} \to \mathbb{R}$ be rd-continuous, $\gamma > 1$ and $\beta = L(s)\gamma$. If

$$|k_1(t, s, p) - k_1(t, s, q)| \le L(s)|p - q|, (p, q) \in \mathbb{R}^{2n}, \quad s < t, \tag{2}$$

$$m = \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_{\beta}(t, a)} \left| F(t) + \int_a^t k_1(t, s, 0)\Delta s \right| < \infty, \tag{3}$$

then the integral equation (1) has a unique solution $z \in C_{\beta}^1(I_{\mathbb{T}}; \mathbb{R}^{2n})$. Let $C_{\beta}^1(I_{\mathbb{T}}; \mathbb{R}^{2n})$ be the linear space of rd-continuous functions such that

$$\sup_{t \in I_{\mathbb{T}}} \frac{\max(|x(t), |x^{\Delta}(t)|)}{e_{\beta}(t, a)} < \infty.$$

We find sufficient conditions for the existence and uniqueness of solution of integral equation (1) in $C_{\beta}^1(I_{\mathbb{T}}; \mathbb{R}^{2n})$.

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ON DISCRETE APPROXIMATION BY THE LERCH ZETA-FUNCTION

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The Lerch zeta-function $L(\lambda, \alpha, s)$, $s = \sigma + it$, with parameters $0 < \alpha \leqslant 1$ and $\lambda \in \mathbb{R}$ is defined for $\sigma > 1$, by Dirichlet series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m + \alpha)^s},$$

and is meromorphically continued to the whole complex plane. It is well known that the function $L(\lambda, \alpha, s)$ for some parameters λ and α is universal in the Voronin sense, i. e., its shifts $L(\lambda, \alpha, s+i\tau)$, $\tau \in \mathbb{R}$, approximate a wide class of analytic functions. For example, this is true for transcendental α. The problem arises does the function $L(\lambda, \alpha, s)$ have some approximation properties for all parameters λ and α. In the case of continuous shifts, this was done in [1]. In the report, we consider the approximation of analytic functions by discrete shifts $L(\lambda, \alpha, s + ikh)$, $k \in \mathbb{N} \cup \{0\}$, $h > 0$. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, and $H(D)$ denote the space of analytic on D functions. Our main result is the following theorem [2].

THEOREM 1. *Suppose that the parameters λ and α, and the number $h > 0$ are arbitrary. Let K be a compact set f the strip D. Then there exists a closed non-empty set $F_{\lambda,\alpha,h} \subset H(D)$ such that, for $f(s) \in F_{\lambda,\alpha,h}$ and every $\varepsilon > 0$,*

$$\liminf_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s \in K} |L(\lambda, \alpha, s + ikh) - f(s)| < \varepsilon \right\} > 0.$$

Moreover, the limit

$$\lim_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s \in K} |L(\lambda, \alpha, s + ikh) - f(s)| < \varepsilon \right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

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SCATTERING OF ELECTROMAGENTIC WAVES WITH SINGLE MEASUREMENT

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The study of electromagnetic waves plays a major role in the inverse scattering theory and antennas. In this research we considering the scattering of fixed frequency electromagnetic from a perfectly conducting flat screen. Fundamentally, this paper is a generalization of [1] for time harmonic Maxwell's equations in the exterior of screen. More accurately, our obstacle is two-dimensional flat screen and interacting incident wave in to three-dimensional space. This type of large and thin object model is admissible in to the study of antennas. Our main results is unique determination of both the supporting hyperplane as well as screen itself, corresponding to the single measurement having non-vanishing electric far-field. The methods we used are based on an analysis of a tangential surface integral equation on the screen. These results hold for arbitrary C^k–screens, $k = 1, 2, 3....\infty$, when they are flat, i.e included in a hyperplane.

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ON SOME FUČÍK PROBLEM WITH ONE BITSADZE-SAMARSKII TYPE NONLOCAL BOUNDARY CONDITION

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Consider the Fučík equation with one Bitsadze-Samarskii type nonlocal boundary condition

$$-x'' = \mu x^+ - \lambda x^-, \tag{1}$$
$$x(0) = 0, \ x(\xi) = 0, \tag{2}$$
$$x(\eta) = 0, \ x(1) = 0 \tag{3}$$

with the parameters $\mu, \lambda \in \mathbb{R}$ and $\xi \in (0, 1]$, $\eta \in [0, 1)$.

The spectrum of the problems (1), (2) and (1), (3) for some ξ and η values is investigated.

The idea of boundary conditions (2) and (3) was taken from the work [1], where the Sturm-Liouville equation

$$-x'' + q(t)x = \lambda^2 x$$

was analyzed (the non-negative real function $q(t)$ has a second piecewise integrable derivatives on the considered interval and λ is spectral parameter).

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APPROXIMATION BY SHIFTS OF DIRICHLET L-FUNCTIONS INVOLVING GRAM FUNCTION

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Let χ be a Dirichlet character modulo $q \in \mathbb{N}$. The Dirichlet L-function $L(s,\chi)$, $s = \sigma + it$, is defined, for $\sigma > 1$, by

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} = \prod_{p} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where the product is taken over all prime numbers, and analytic continuation elsewhere.

Let t_τ, $\tau > 0$, be the solution of the equation $\theta(t) = (\tau - 1)\pi$, where $\theta(t)$, $t > 0$, is the increment of the argument of the function $\pi^{-s/2}\Gamma(s/2)$ along the segment connecting the points $s = 1/2$ and $s = 1/2 + it$. t_τ is called the Gram function.

In the report, we consider joint approximation of analytic functions by shifts $L(s + it_\tau^\alpha, \chi)$ of Dirichlet L-functions. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$ the class of continuous non-vanishing functions on K that are analytic in the interior of K. The main result is the following universality theorem.

THEOREM 1. [1]. *Suppose that $\alpha_1, \ldots, \alpha_r$ are fixed different positive numbers, and $\chi_1, \ldots, \chi_r$ are arbitrary Dirichlet characters. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |L(s + it_\tau^{\alpha_j}, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Moreover, the limit

$$\lim_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |L(s + it_\tau^{\alpha_j}, \chi_j) - f_j(s)| < \varepsilon \right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Also, approximation by some compositions $\Phi(L(s + it_\tau^{\alpha_1}, \chi_1), \ldots, L(s + it_\tau^{\alpha_r}, \chi_r))$ will be discussed.

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EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS FOR A THIRD-ORDER BOUNDARY VALUE PROBLEM WITH NONLOCAL CONDITIONS

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We study the existence of multiple positive solutions for a nonlinear third-order differential equation subject to various nonlocal boundary conditions. The boundary conditions that we study, contain Stieltjes integral and include the special cases of m-point conditions and integral conditions. The main tool in the proof of our result is Krasnosel'skii's fixed point theorem. To illustrate the applicability of the obtained results, we consider examples.

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BIPOLAR LATTICES AS CODOMAINS FOR FUZZY EXTENSIONS OF L-TOPOLOGIES

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Recall that given a complete lattice $(L, \leq, \wedge, \vee)$ with lower $\mathbf{0}$ and upper $\mathbf{1}$ bounds, an L-topology on a set X is a family $\tau \subseteq L^X$ containing constant functions $\mathbf{0}_X$, $\mathbf{1}_X$, and closed under taking finite meets and arbitrary joins [1],[2]. Given a complete lattice $(M, \leq_M, \wedge_M, \vee_M, \mathbf{0}_M, \mathbf{1}_M)$ (probably different from L, an (L, M)-fuzzy topology on a set X is a mapping $\mathcal{T} : L^X \to M$ such that $\mathcal{T}(\mathbf{0}_X) = \mathcal{T}(\mathbf{1}_X) = \mathbf{1}_M$, $\mathcal{T}(A \wedge B) \geq A \wedge B$ for every $A, B \in L^X$ and $\mathcal{T}(\bigvee_i A_i) \geq \bigwedge_i \mathcal{T}(A_i)$ for every $\{A_i \mid i \in I\} \subseteq L^X$ [3], [4].

Let $\mathcal{L} = L^+ \times L^-$ where $L^+ = L$ and L^- is a copy of L with reversed order, i.e. $a \leq b, a, b \in L^+ \implies -a \geq -b, -a, -b \in L^-$. By setting $(a, b) \preceq (a', b') \iff a \leq a', b \geq b'$ we introduce a partial order $\preceq$ on $\mathcal{L}$. In the result $(\mathcal{L}, \preceq)$ becomes a complete lattice that we interpret as a bipolar lattice with lower $(\mathbf{0}, -\mathbf{1})$ and upper $(\mathbf{1}, -\mathbf{0})$ bounds and with the "zero element" $(\mathbf{0}, -\mathbf{0})$.

In this talk we present a model allowing to extend an L-topology $\tau \subseteq L^X$ to a full-bodied $(L, \mathcal{L})$-fuzzy topology $\mathcal{T} = (\mathcal{O}^+, \mathcal{O}^-) : L^X \to \mathcal{L}$ where $\mathcal{O}^+(A) \in L^+$ determines the degree of openness of a fuzzy set A and $\mathcal{O}^-(A) \in L^-$ the degree of its non-openness. This model is based on the possible additional use of a binary conjunction-type operation $* : L \times L \to L$ (in particular $*$ may be lower semi-continuous t-norm or just $* = \wedge$, see [5]). The choice of the operation $*$ allows to specify different additional properties of mappings $\mathcal{O}^+ : L^X \to L^+$ and $\mathcal{O}^- : L^X \to L^-$. In particular, if $*$ is the Łukasiewicz t-norm, then $\mathcal{O}^-(A) = -\mathcal{O}^($A)$ for each $A \in L^X$ and this well corresponds with the intuitive interpretation of existence and not existence degrees of a certain property (in our case, the degrees of a fuzzy set to be open).

In conclusion, we shall sketch the ideas of application of such models for the study of other issues of fuzzy topology as well as the prospects for similar, based on bipolar lattices models for the study of other different fuzzy mathematical structures.

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ASYMPTOTIC ANALYSIS OF STURM–LIOUVILLE PROBLEM WITH NONLOCAL BOUNDARY CONDITIONS

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Consider the following one-dimensional Sturm–Liouville equation

$$-u''(t) + q(t)u(t) = \lambda u(t), \quad t \in [0, 1], \tag{1}$$

where the real-valued function $q \in C[0, 1]$.

Sturm–Liouville Problem (SLP) which consist of equation (1) with one classical (local) boundary condition $u(0) = 0$ [3] or $u'(0) = 0$ [1] and another two-point Nonlocal Boundary Condition (NBC)

$$\text{(Case 1)} \qquad u'(1) = \gamma u(\xi), \qquad \xi \in [0, 1], \tag{2_1}$$
$$\text{(Case 2)} \qquad u'(1) = \gamma u'(\xi), \qquad \xi \in [0, 1), \tag{2_2}$$
$$\text{(Case 3)} \qquad u(1) = \gamma u(\xi), \qquad \xi \in [0, 1), \tag{2_3}$$

where $\gamma \in \mathbb{R}$ were considered.

We obtained asymptotic formulas for eigenvalues and eigenfunctions for these two cases. Now we investigate more general case with Robin type boundary condition $\cos \alpha \, u(0) + \sin \alpha \, u'(0) = 0$, $\alpha \in (0, \pi)$ [2] and NBC (2).

Finally we discussed how to get asymptotic expansions for real eigenvalues in the case more complicated NBC.

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FINITE DIFFERENCE METHOD FOR TWO-DIMENSIONAL ELLIPTIC EQUATION WITH THE MULTIPLE INTEGRAL IN NONLOCAL CONDITION

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We consider finite difference approximation to the solution of following boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y, u), \quad (x, y) \in \Omega = \{0 < x, y < 1\},$$

$$u|_{\partial\Omega} = \iint_{\Omega} K(x, y)u(x, y)dxdy.$$

The main aims of our study are the convergence of finite difference scheme and the iterative methods for solution of the system of nonlinear difference equations.

Motivated by some theoretical results of [1] and [2] we study the spectrum structure of the corresponding difference eigenvalue problem.

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AN APPLICATION OF SPH TO COMBUSTION

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We present a simulation technique for the system of compressible Navier-Stokes equations, barotropic gas pressure-density law, heat equation and convection-reaction-diffusion equations with Arrhenius kinetics in Lagrange coordinates:

$$T_t = D\triangle T + c_1 c_2 Q e^{-E/k_B T},$$

$$c_{i,t} = D_i \triangle c_i - c_1 c_2 k e^{-E/k_B T}, \quad i \in \{1, 2\},$$

$$v_t = -\frac{1}{\rho}\nabla P + \nu\triangle v + \beta(T - T_0),$$

$$P = P_0 \rho^\alpha, \quad \rho_t = -\text{div}(\rho v),$$

where T is temperature field, c_1 is the concentration of fuel, c_2 the concentration of oxygen, v is the gas velocity field, P is pressure field, ρ density, D is the heat diffusion coefficient, D_1 and D_2 the diffusion coefficients of the fuel and oxygen, ν viscosity, β thermal expansion coefficient.

We develop and implement a smoothed particle hydrodynamics (SPH) method [1] for this system of equations with various boundary conditions. The kernel function is a Gauss kernel

$$W(x, h) = (h\sqrt{\pi})^{-d} exp(-x^2/h^2),$$

cut off at $|x| = 3h$, so the Laplace operator is approximated as

$$\triangle f(X_i) \approx 2 \sum_j \frac{m_j}{\rho_j} f(X_j) \frac{X_i - X_j}{|X_i - X_j|^2} \nabla W(X_i - X_j, h).$$

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ON MATHEMATICAL MODELLING, SIMULATION AND OPTIMIZATION OF SUPERCRITICAL CO2 EXTRACTION FLOW

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Supercritical carbon dioxide extraction is a process of extracting valuable compounds from a wide range of materials using CO2 in its supercritical state. Authors consider mathematical models for 3D flow in extraction of biologically active compounds from plants. Similar models and treatment of experimental data for the process also considered in [1], [2].

The governing equations are

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) = -\nabla p + \frac{1}{Re}\triangle \mathbf{v} + \mathbf{f},$$

$$\frac{\partial \varphi}{\partial t} + \nabla \cdot (\mathbf{v}\varphi) = \alpha\triangle \varphi + \mathbf{R}(\varphi, \psi),$$

where $\varphi(r, t)$ is the concentration of the active compound in the solvent and $\psi(r, t)$ is the concentration of the active compound bounded in the intact cells.

These are discretized by finite volume techniques. We discuss parametrization of the process as well and some techniques how to perform the optimization process. Moreover we present the graphical results of obtained fields and extraction curves and discuss main aspects of several underlying modelling stages.

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CITATION INDICES DEPENDENT ON TIME

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Citation analysis is a globally used tool for evaluating the scientific performance and impact of various subjects, including scholars (scientists, researchers), journals, etc. Several citation indices have been introduced for this purpose. Certainly the most considered index in this area is the Hirsch index H, see [2]. Almost immediately after the introduction of the h-index H, several complaints appeared. Subsequently, a number of its modifications were published. Among the other used citation indices, let's us mention g-index of Egghe G, see [1] and Kosmulsky's MAXPROD index, [3]. Most citation indices (including those already mentioned ones) do not distinguish the publication time of papers of the considered author and are based on a simple characterization of each author by a decreasing vector $\mathbf{a}$ only. Note that the vector of citation is defined as $\mathbf{a} = (a_1, \ldots, a_n)$, $a_1 \geq \cdots \geq a_n$, where n is the number of published papers of author A (in the considered database) and a_i denotes the number of citations obtained by i-th most cited article. Then scholars who have no papers or citations in recent decades will retain their (possibly high) (static) citation index.

Note that there were several attempts to introduce time-dependent (generalization of) h-index. We will use a different approach when describing the time-dependent author's performance, where not only the total number of citations of the involved papers is taken into account (the vector $\mathbf{a}$), but also the time of publication, where a vector $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{N}^n$ is considered. Here y_i is the year age of the i-th paper possessing a_i citations.

Consequently, we exploit the fact that the h-index and MAXPROD can be considered as Sugeno and Shilkret integrals with respect to the counting measure, respectively. Note that the g-index cannot be represented by some integral. To introduce new time-dependent (dynamic) indices, we propose 3 ways to modify the mentioned citation indices. In the first one, we replace the input vector $\mathbf{a}$ by an reordered vector $\left(\frac{a_1}{y_1}, \ldots, \frac{a_n}{y_n}\right)$ (this approach can be applied to arbitrary indices, not only represented by an integral), in the second one we change the counting measure to a measure dependent on the vector $\mathbf{y}$, and finally we combine the two previous ones.

We believe that our approach is more fair for the scientific beginners and for all scholars active in the field. More, our new dynamic indices allow to compare authors with the same or similar static indices in a way where active scientists are ranked higher. We will show the advantages of these new dynamic indexes on examples.

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ON UNIVERSALITY OF PERIODIC ZETA-FUNCTIONS

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Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ be a periodic with period q sequence of complex numbers, and $s = \sigma + it$ a complex variable. The periodic zeta-function $\zeta(s; \mathfrak{a})$ is defined, for $\sigma > 1$, by the series $\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} a_m m^{-s}$, and has a meromorphic continuation to the whole complex plane.

Suppose that the sequence $\mathfrak{a}$ additionally is multiplicative, i. e., $a_{mn} = a_m a_n$ for all $(m, n) = 1$, and $a_1 = 1$. For example, a Dirichlet character modulo q is multiplicative. Then it is known [1] that the function $\zeta(s; \mathfrak{a})$ is universal in the sense that the shifts of $\zeta(s + i\tau; \mathfrak{a})$, $\tau \in \mathbb{R}$, approximate a wide class of analytic functions.

In the report, we consider the joint universality of periodic zeta-functions with multiplicative coefficients. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, $\mathcal{K}$ be the class of compact sets of the strip D with connected complements, and $H_0(K)$, $K \in \mathcal{K}$, the class of continuous non-vanishing functions on K that are analytic in the interior of K. Denote by $U_1(T_0)$ the class of real increasing to $+\infty$ continuously differentiable functions $\gamma(\tau)$ with monotonic derivative $\gamma'(\tau)$ on (T_0, ∞) such that $\gamma(2\tau) \max_{\tau \leqslant u \leqslant 2\tau} (\gamma'(u)^{-1}) \ll \tau$ as $\tau \to \infty$. Then we have

THEOREM 1. [2]. *Suppose that the sequences $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ multiplicative, $a_1, \ldots, a_r$ are real algebraic numbers linearly independent over $\mathbb{Q}$, and $\gamma(\tau) \in U_1(T_0)$. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |\zeta(s + ia_j\gamma(\tau); \mathfrak{a}_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Also, the joint approximation by shifts $(\zeta(s + i\gamma_1(\tau); \mathfrak{a}_1), \ldots, \zeta(s + i\gamma_r(\tau); \mathfrak{a}_r))$ will be considered. In this case, we require that the functions $\gamma_j(\tau)$ satisfy $\gamma_j(\tau) = \widehat{\gamma}_j(\tau)(1 + o(1))$, where $\widehat{\gamma}_j(\tau) = o(\widehat{\gamma}_{j_0}(\tau))$ for some j_0.

Moreover, we discuss the joint discrete approximation of analytic functions by shifts $(\zeta(s + ih_1\gamma_1; \mathfrak{a}_1), \ldots, \zeta(s + ih_r\gamma_r; \mathfrak{a}_r))$, where $\{\gamma_k : k \in \mathbb{N}\}$ is the sequence of imaginary parts $\gamma_k > 0$ of non-trivial zeros of the Riemann zeta-function.

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CONSTRUCTION OF SOLUTIONS TO CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS VIA OPERATOR TECHNIQUES

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Fractional differential equations (FDEs) have emerged in recent decades as an important class of models that can be used to describe various real-world phenomena. These equations have been successfully used in a plethora of fields ranging from physical applications [1], health research [2] and even finance [3].

In this study, the solutions y to FDEs are represented as power series with fractional basis elements $y = \sum_{j=0}^{+\infty} c_j x^{j/n}$, where the fractional differentiation order is $\alpha = \frac{1}{n}, n \in \mathbb{N}$. This concept of solutions to FDEs allows the consideration of equivalency between FDEs of a specific form and respective ordinary differential equations (ODEs) that are obtained via a nonlinear independent variable transformation. However, the aforementioned transformation can only lead to a computational scheme if a certain transfer function is approximated via a series truncation.

Mathematically, the scheme can be described as follows. Consider the following FDE:

$$^C\mathbf{D}^{(1/n)}y = F(x,y),$$

where $F(x,y)$ is an arbitrary function. Firstly, this FDE is transformed into:

$$\left({}^C\mathbf{D}^{(1/n)} \right)^n y = G(x,y) + \Psi(x),$$

where $G(x,y)$ and $\Psi(x)$ depend on the form of $F(x,y)$. The function $\Psi(x)$ is given by power series with coefficients defined via a recurrence relation. Finally, the transformation $t = \sqrt[n]{x}$ yields:

$$\frac{\mathrm{d}\widehat{y}}{\mathrm{d}t} = H\left(\widehat{y}, t\right), \quad y = y(x) = \widehat{y}\left(\sqrt[n]{x}\right).$$

This ODE can be analyzed via conventional techniques and its solution transformed into the FDE solution using the inverse transformation.

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SPECTRUM PROPERTIES OF THE FUČÍK PROBLEM WITH NONLOCAL BOUNDARY CONDITION

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Let us analyze the Fučík problem with nonlocal two-point condition in the right side boundary

$$x'' = -\mu x^+ + \lambda x^-, \tag{1}$$

$$x(0) = 0, \tag{2}$$

$$x(1) = \gamma x(\xi), \quad \gamma \in \mathbb{R}, \tag{3}$$

here $\xi = \frac{m}{n} \in (0,1)$, m and n $(0 < m < n)$ are positive coprime integer numbers. The spectrum of the Fučćk problem with classical boundary conditions are quite well analyzed. However, in case of nonlocal boundary conditions the spectrum can differ considerably. Thus, the aim of investigation is to analyze main properties of the nonlocal problem spectrum and compare them with classical Fučćk spectrum. The analytical description and visualization of the spectrum will be provided.

Some of the results are the logical continuation and generalization of previous authors' investigations. This investigation continues and generalize the results obtained for problem (1) – (3) with $\xi = \frac{1}{2}$ [1], $\xi = \frac{1}{3}$ [2], $\xi = \frac{1}{4}$ [3] and $\xi = \frac{1}{n}$ [4].

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ADVANTAGES OF THE SMOOTHLY TRIMMED MEAN

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L-estimators, defined as a linear combination of order statistics, are commonly used for robust inference of the central tendency. The trimmed mean is one of the most popular L-estimators, among others. In 1973 M. Stigler [1] showed that it has a significant disadvantage the limiting distribution of the trimmed mean may not be asymptotically normal if the data have a discrete or continuous distribution with gaps. In his paper M. Stigler proposed to use the smoothly trimmed and showed that the asymptotic distribution of such statistics is asymptotically normal for any distribution.

We propose a slightly more general version of smoothly trimmed mean, derive its variance and establish the empirical likelihood method for this estimator. There are many statistical procedures which are based on the classical trimmed mean estimator (see, for example, the classical book by Wilcox [2]). In this work for different testing procedures we simply replace the classical trimmed mean estimator by the smoothly trimmed mean estimator and compare them by extensive simulation analysis. Some applications for real data examples will be analysed as well.

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ON JOINT UNIVERSALITY IN THE SELBERG-STEUDING CLASS

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The Selberg class $\mathcal{S}$ is defined axiomatically and consists of Dirichlet series satisfying four following axioms: the Ramanujan hypothesis, an analytic continuation, functional equation, multiplicativity. The Selberg-Steuding class $\mathcal{S}^*$ is a complemented Selberg class by an arithmetic hypothesis related to distribution of prime numbers and an existence of the Euler product.

The universality property for the Riemann zeta-function $\zeta(s)$ (which is a member of $\mathcal{S}$ also) was obtained by S.M. Voronin in [3], while the first universality result related to the class $\mathcal{S}^*$ was obtained by J. Steuding in [2].

In the report, we present the joint universality theorem for L-functions from the Selberg - Steuding class of functions belonging to the class $\mathcal{S}$ and satisfying the arithmetic condition on the distribution of primes (denote this fact $\mathcal{S}_1$). More precisely, we discus simultaneous approximations of a collection of analytic functions $\big(f_1(s), \ldots, f_r(s)\big)$ in the strip $D_L = \{s \in \mathbb{C} : \sigma_L < \sigma < 1\}$ by a collection of shifts $\big(L(s + ia_1\tau), \ldots, L(s + ia_r\tau)\big)$, $L(s) \in \mathcal{S}_1$, where $\sigma_L > \frac{1}{2}$ is a certain number depending on L.

Denote by $\mathcal{K}_L$ the class of compact subset of the strip D_L with connected complements, and by $H_{0L}(K)$, $K \in \mathcal{K}_L$, the class of continuous non-vanishing functions on K that are analytic in the interior of K. The main result is following theorem.

THEOREM 1. *Suppose that $L(s) \in \mathcal{S}_1$, and real algebraic numbers $a_1, \ldots, a_r$ are linearly independent over the field of rational numbers $\mathbb{Q}$. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}_L$ and $f_j(s) \in H_{0L}(K_j)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0, T] : \sup_{1 \le j \le r} \sup_{s \in K_j} |L(s + ia_j\tau) - f_j(s)| < \epsilon\right\} > 0.$$

Moreover, "$\liminf$" can be replaced by "$\lim$" for all but at most countably many $\epsilon > 0$.

The content of presentation is based on the joint work by R. Kačinskaitė, A. Laurinčikas and B. Žemaitienė [1].

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ESTIMATION OF THE SOLUTIONS OF A MULTI-OBJECTIVE LINEAR PROGRAMMING PROBLEM

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Multi-objective linear programming (MOLP) is a type of mathematical optimization technique used to solve problems that involve multiple conflicting objectives. In MOLP, the goal is to find a set of solutions that simultaneously optimize two or more objectives, subject to a set of linear constraints. MOLP problems are used in various fields, including engineering, economics, finance, and environmental management. They are particularly useful in situations where decision makers must balance multiple objectives.

When solving multi-objective linear programming problems using Zimmermann's objective functions [2] or fuzzy orderings [1], it can be seen, that the solutions tend to be *close* to the centroid or mass center of individual solutions set. In this case, we define proximity by metrics, which shows how far the point is from the individual solution in terms of an objective function values. In this work, the observation that the optimal solutions are *around* the centroid is looked at in details. Intuitively, the question arises whether it is true that

$$d(x; c_j) \geq d(x; c_M),$$

where x is an optimal solution, c_j are individual solutions, c_M is a centroid and d is a global metric or distance-like function which takes into account all the metrics described above. Thus, this work is devoted to finding the necessary assumptions for the inequality to be true, and finding the best way to determine the point c_M.

The inequality can be used as a rough estimate of the region where the optimal solution for a given MOLP problem can be found. It is useful when the optimisation is done using an iterative algorithm. It allows to start iterations closer to the optimal solution, by choosing the point c_M or points around it as the starting points, thus expecting, that the algorithm will converge faster.

REFERENCES

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